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AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

Founded in 1895 by GEORGE E. HALE *and* JAMES E. KEELER

Edited by

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Ryerson Physical Laboratory of the University of Chicago

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OTTO STRUVE
Yerkes Observatory of the University of Chicago

NOVEMBER 1935

CHARLES EDWARD ST. JOHN - - - - - - - - - - - - - *C. G. Abbot* 273

KINEMATICS AND WORLD-STRUCTURE - - - - - - - - - *H. P. Robertson* 284

TWO METHODS OF INVESTIGATING THE NATURE OF THE NEBULAR RED-SHIFT
Edwin Hubble and Richard C. Tolman 302

A QUANTITATIVE STUDY OF CERTAIN PHASES OF F-TYPE SPECTRA - *J. A. Hynek* 338

REVIEWS
The Binary Stars, ROBERT GRANT AITKEN (G. V. B.), 368

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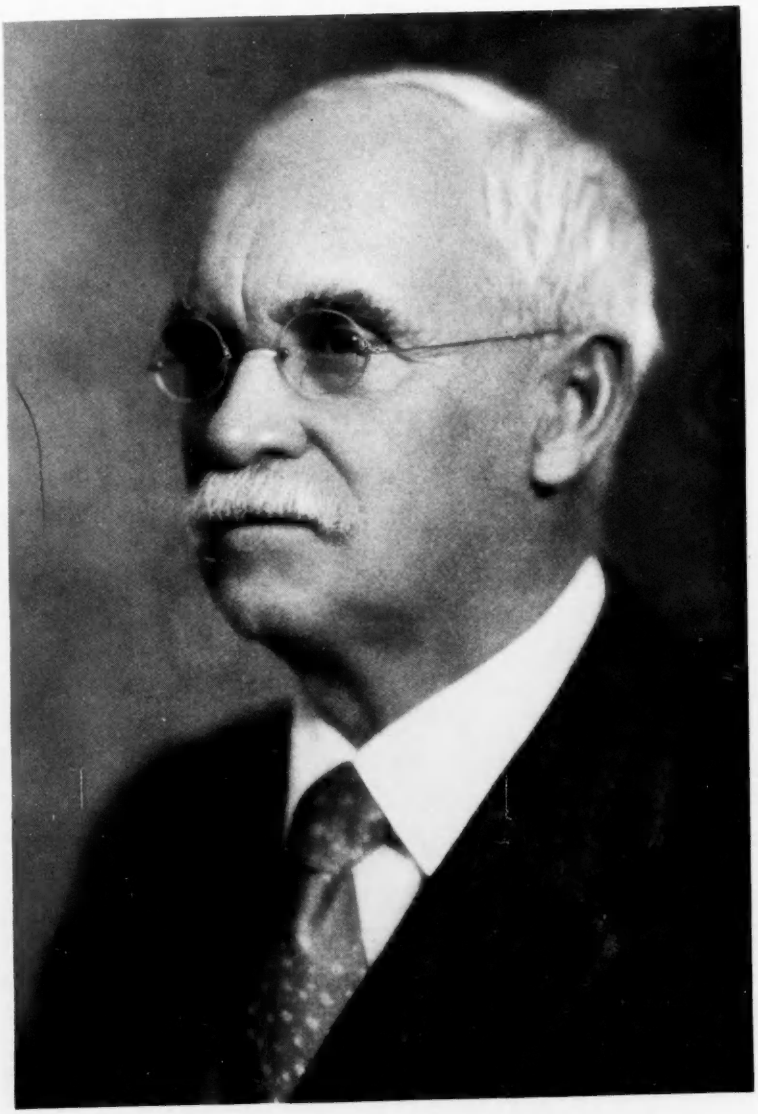

CHARLES EDWARD ST. JOHN

1857–1935

THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 82 NOVEMBER 1935 NUMBER 4

CHARLES EDWARD ST. JOHN

By C. G. ABBOT

Dr. Charles Edward St. John closed his useful life on April 26, 1935. Devoted and successful in his astrophysical work, he yet found precious and abundant resource in birds and nature and literature. Almost more than for research he had a passion for friends. It was therefore entirely in character that, undismayed as the end approached, he gathered his closest friends about him, one by one, gave them mementos from among his long-cherished books and familiar possessions, and with good cheer bade them affectionate farewells.

St. John was born at Allen, Michigan, March 15, 1857, the youngest of five children, three sons and two daughters. Their mother, née Lois Bacon, born in Saugus, New York, was a woman of excellent poise and human understanding, with a natural intellectual equipment above the average, to which was added whatever of education was available in Michigan between 1825 and 1840. Their father, an able and successful millwright, designed and constructed many lumber and grist mills, but a head injury which he had suffered occasioned him considerable mental and physical distress.

The two daughters, St. John's sisters, were intellectually gifted, and had successful business and teaching careers. One brother of much promise died at thirty. The other, before completing his education, enlisted in the Civil War for three years in an artillery regiment. After his return he engaged in millwork for some years, and then, notwithstanding the handicaps of meager education and an

accident which involved the loss of his right hand, he studied law and became an outstanding attorney in southern Michigan.

At nineteen years of age Charles Edward St. John was graduated from the Michigan Normal School, the youngest in his class. He had overtaxed himself, and suffered a long period of ill health. At twenty-nine he was at last able to take an instructorship in physics at the Normal School, and only at thirty-one received the degree B.S. from Michigan Agricultural College. After postgraduate work in magnetism and electricity, being then thirty-five years of age, he received the degree of Master of Arts from Harvard University, and was awarded the John Tyndall Fellowship, which included a year of study abroad. He spent this at Heidelberg, Germany, where he devoted himself principally to the then outstanding branch of physics, the study of black-body radiation.

St. John received his Ph.D. degree from Harvard in 1896, and after a year's instructorship at the University of Michigan, he accepted the professorship of physics and astronomy at Oberlin. He remained at Oberlin eleven years, becoming dean of the college of arts and sciences. It was his habit to spend vacation times at the Yerkes Observatory, where, about 1900, he assisted Dr. E. F. Nichols in his remarkable pioneering measurements of the heat of the stars with the radiometer. It was here that he acquired among intimate friends the name "Heiliger," which remained with him through life. At the Yerkes Observatory St. John became intimate with Dr. George E. Hale and other members of the staff, through whom, after their removal to Mount Wilson, St. John obtained the opportunity for the twenty-two years of research on which his reputation mainly rests.

When, at more than fifty years of age, in 1908, St. John relinquished teaching to devote himself to research at Mount Wilson, it was an exceptionally bold step. Indeed, for the first one or two years his work gave little promise of the independence and importance it was soon to assume.

Those were happy days on Mount Wilson, before the completion of the great telescopes, when all the observations except Barnard's (who scarcely ever slept anyway) were solar and everybody else was through work in the evening. Dr. Hale was then actively designing

such unique things as the tower telescopes, was discovering magnetism in sun-spots, and was the life of the party at meal times and when we sat about a roaring fire in the evening in the old Monastery. Though not himself, as a rule, one of the principal entertainers, St. John, whenever present, was one of the most appreciative of the entertained. In that company were W. S. Adams, H. G. Gale, F. Ellerman, E. E. Barnard, J. C. Kapteyn, and others; new things were being discovered daily, and enthusiasm reigned. Those were the days before the invention of bridge, when duplicate whist was the aristocrat among card games. St. John used to join in the hands, but not very successfully. It was necessary to use sixteen decks of cards lest Adams remember every former play. We had also a billiard table on the mountain, and St. John took lessons for a time, but was discouraged by his poor success in playing the game as it should be. Not willing to bang around the table like the rest of us, he soon gave it up.

Dr. Kapteyn had related to me an experiment in telepathy which he and another professor of the University of Groningen had made with negative results. They sat back to back, and after moments of concentrated attention one wrote on his pad a digit, saying "Ready." The other then wrote a digit that came first to mind. After several hundred numbers were thus written, they compared the number of agreements with the number of expected agreements according to the theory of probability. They found no excess over expectation.

The experiment interested me, and one evening when the computers had a party on Mount Wilson I proposed it as a game. One of the party went out into another room, while the rest concentrated on the numbers. In most cases we got negative results. But in both cases when St. John was out, as well as in both cases when Miss Wilhoit, Dr. Hale's secretary, was out, decidedly positive results occurred, particularly so with St. John. I should have liked to continue the experiments, but they were too dull for the party. If, indeed, one mind may influence another without employing ordinary means of communication, it would not be surprising if St. John, with his quick mental reactions and high-strung temperament, might have been especially sensitive to telepathic communication.

St. John's researches and publications are numerous and impor-

tant. He had the merit of thoroughly reducing to publication the investigations that he undertook, so that with the exception of a considerable part of his work on the solar rotation, which still remains unpublished, we have his results in available form. Most of his researches depended on exact wave-lengths, and concerned the interpretation of minute deviations in position, or distortions of form of spectral lines. Among his earliest work undertaken at Mount Wilson was the investigation of the H and K lines of calcium,[1] of which the prevalence and enormous elevation in the chromosphere had been an enigma for half a century. Indeed, they remained enigmatic until a little later, when the progress of physics disclosed the influences of ionization potentials, atmospheric density, and other conditions.

In the first of the three papers St. John laid a sound basis by measuring the absolute wave-lengths of H and K in terrestrial sources in terms of the then recent interferometer standards of Fabry and Buisson. St. John used three spectroscopes of different resolving powers. He transferred the Fabry and Buisson standards through the spectrum of iron to the calcium spectrum produced in the arc with plentiful and meager calcium supply, in the spark and in the electric furnace. The results were in close accord, and the error of the wave-lengths he deduced was not above 0.001 A.

In the second paper St. John applied this highly exact basis to the study of the positions of calcium lines in the solar spectrum. The work has characteristic thoroughness. To mention his results briefly in his own words:

> Calcium vapor producing the absorption line K_3 in the solar spectrum has a [mean] descending motion over the general surface of the sun of 1.14 km. per second . . . . the bright emission line K_2 is due to an ascending motion . . . . of 1.97 km. per second. . . . . The angular velocity of the high-level calcium vapor producing the absorption line K_3 is nearly constant for the latitudes of observation, being 15°.5 and 15°.4 per day at latitudes 6°.6 and 38°.4 respectively. . . . . The wave-lengths of K_2 and K_3 reduced to the limb [agree so closely with] the corresponding wave-length in the arc [that] the mean pressure in the intermediate emitting layer [must be] approximately one atmosphere. . . . . [There is no evidence] for positions of intermediate and highest levels of calcium of currents of appreciable velocity parallel to the solar surface. . . . . The widths of H_3 and K_3 at the center [and near] the limb . . . . compared to the corresponding widths

[1] *Mt. W. Contr.*, Nos. 44, 48, 54; *Ap. J.*, **31**, 143, 1910; **32**, 36, 1910; **34**, 57, 131, 1911.

in the arc point to an extremely small quantity of calcium vapor in the upper levels of the solar atmosphere. . . . . The [observations] indicate a relatively dense and thick layer of emitting vapor as compared with the absorbing layer. . . . . The average appreciable height of the chromospheric calcium . . . . is about 5,000 km. above the photosphere.

In the third paper St. John applied similar methods of research to discover "the motion and condition of calcium vapor over sun-spots and other special regions." Here again we must admire the devotion of the author to thoroughness and the securing of abundant evidence. He concludes:

In the great majority of sun-spots the calcium vapor is descending in the umbrae with velocities varying from 0.68 to 2.2 km. per second . . . . [while] over the penumbrae [though] the source of the bright line K_2 has little if any vertical motion, the calcium vapor producing the absorption line is descending with approximately the same mean velocity as over the general disk. . . . . Over flocculi [near sun-spots] the emitting vapor shows [doubtfully] upward motion, ... [while] the absorbing vapor . . . . is descending with the same mean velocity as over the general disk. . . . . Radial motion of the calcium vapor inward across the penumbrae is shown; . . . . [also] rotary motion [occasionally]. . . . . [These motions] result in a spiral or a vortical motion converging upon the umbra. . . . . The relative intensities of the K_1, K_2, and K_3 lines [depend] on great differences in their radiation coefficients and consequent different depths.

The preoccupation of St. John with researches whose results depended on knowledge of exact wave-lengths in terrestrial and solar spectra led to his election to the presidency of the commission of the International Astronomical Union dealing with that subject. International co-operation, begun under the auspices of the Union for Co-operation in Solar Research in 1907, had led step by step from the primary standard of the wave-length of the red cadmium line through a list of secondary and tertiary standards depending on the most exact applications of interferometry. With this fundamental basis, St. John and his collaborators at the Mount Wilson Observatory worked through and published in 1928 the great *Revision of Rowland's Preliminary Table of Solar Spectrum Wave-Lengths with an Extension to the Present Limit of the Infra-Red.* This monumental work gives: (1) the wave-lengths on the I.A. system; (2) the revised and extended identifications with elements and compounds; (3) the intensities; (4) new and important data convenient for physical and

astrophysical investigations, including relative intensities of lines in sun-spots and on the disk, temperature classification from furnace spectra, classification from behavior under pressure, in the explosive spark, and in the high current arc; and, finally, the excitation potential of all lines whose multiplet designations were then known. The tedious measurements and prolonged research into literature by St. John himself in bringing this indispensable revision to completion and publication are alone sufficient to give him a permanent place in the history of astrophysics. Important contributory researches by St. John alone, or with co-authors, which were needful steps in this indispensable investigation, are found in the *Contributions from the Mount Wilson Observatory* as follows: Nos. 61, 1912; 75, 1913; 106, 1915; 120, 1916; 137, 1917; and 202, 1921.[2]

In a pair of especially interesting papers[3] entitled "Radial Motion in Sun-Spots," St. John confirms and amplifies Evershed's discovery of differential velocity displacements of Fraunhofer lines in the penumbrae of sun-spots, indicating an outwardly directed radial movement of the gases of the reversing layer and their subsequent flow tangential to the surface of the sun. St. John's work extended over two years, 1910–1912, during which he investigated the velocity displacements of a host of Fraunhofer lines representing elements of a great range of atomic weights. As a result he gives in the first paper a remarkable figure which illustrates the flow of gases at different levels in the penumbrae of sun-spots. The sun-spot is shown to be a vortex analogous to a terrestrial tornado or hurricane:

> There is a whirling upward rush of material from the interior of the sun, which spreads out radially with rapidly decreasing velocity, tangential to the solar surface, and entrains with it the gases of the reversing layer. The actual vortex is deep seated, the outflow into the reversing layer being a portion of the upper part of it, the inflow from the chromosphere being a secondary effect, a superficial indication of the underlying vortex in which the magnetic field originates.

In the second paper St. John further amplifies conclusions regarding the levels of different chemical elements in the sun, and discusses the velocity of solar rotation and of displacement of gases in sun-spots in relation to levels.

[2] *Ap. J.*, **36**, 14, 1912; **39**, 5, 1914; **42**, 231, 1915; **44**, 15, 1916; **46**, 138, 1917; **53**, 260, 1921.

[3] *Mt. W. Contr.*, Nos. 69, 74; *Ap. J.*, **37**, 322, 1913; **38**, 341, 1913.

From these studies he very naturally passed[4] to a comparison of these results on levels with those derived from S. A. Mitchell's remarkable flash spectrum which was observed in 1905.

> The writer finds no better . . . . summing up . . . . than that [concluding] the discussion of the displacements in the penumbrae of sunspots. "The resulting distribution shows that the H_3 and K_3 lines of calcium are the lines of highest level, followed by the $H\alpha$ line of hydrogen, and that, in the main, the heavy and rare elements occur in detectable amounts only in the lower portions of the solar atmosphere."

In two careful papers of about this same period[5] St. John shows that the solar spectral evidence does not on the whole support Professor Julius' belief in a prevailing importance of anomalous dispersion in the sun.

What perhaps St. John himself considered his most important investigations began to be published in 1917 on "The Principle of Generalized Relativity and the Displacement of Fraunhofer Lines Toward the Red." His first paper[6] reached the negative conclusion that "within the limits of error there is no evidence in these observations of a displacement to longer wave-lengths, either at the center or at the limb of the sun, of the order of 0.008 A, as required by the principle of relativity." These measurements were made by St. John on 40 lines in the band spectrum of nitrogen (cyanogen) "selected with a view to precision of measurement and freedom from blends" and for which pressure shifts are negligible.

Later investigations indicated that pressure is of but insignificant importance in the solar atmosphere accessible to spectroscopic measurements, instead of being of the order of 5 to 7 atmospheres as was supposed in 1917. Meanwhile, the importance of the theory of relativity and its acceptance by physicists steadily increased. As early as 1923 St. John had become satisfied that the solar spectrum does give evidence of the influence of gravitational spectral displacements as required by the general theory of relativity.[7]

[4] *Mt. W. Contr.*, No. 88; *Ap. J.*, **40**, 356, 1914.

[5] *Mt. W. Contr.*, Nos. 93, 123; *Ap. J.*, **41**, 28, 1915; **44**, 311, 1916.

[6] *Mt. W. Contr.*, No. 138; *Ap. J.*, **46**, 249, 1917.

[7] *Mt. W. Comm.*, No. 96; *Proc. Nat. Acad.*, **12**, 65, 1926.

His final publication on the subject was made in 1928.[8] In this paper St. John outlines the effect to be expected, the sufficient precision of solar spectral measurements to detect it, the absence of the pressure effect formerly so much feared as a hindrance, the expectation of differing results depending on solar levels and wave-lengths, the degree of confusion to be expected from the influences of motion in the solar gases. He goes on with measurements on the positions of 497 iron lines of groups *a* and *b*, 89 iron lines of groups c_5 and d_5, 18 similar manganese lines, 515 closely spaced lines among 3883 found in the band of cyanogen, including the former 40, 6 lines of silicon, 10 high-level lines of calcium, sodium, manganese, and hydrogen, and 402 lines of titanium, making in all 1537 lines, for which he compares the solar and arc spectra in vacuum.

St. John gives us the following pleasantry:

> My original investigation was confined to some 40 lines and gave negative results. In view of later work on the complete band, these lines might be called the "Forty Thieves." . . . . The lines in my original paper are included among the 515 lines. Their remeasurement agrees well with the original measures, which failed to show displacements to the red in agreement with the Einstein theory of gravitation. Their influence, however, is counteracted in the final mean, based on a far greater number of lines.

His conclusion is:

> Lines originating at a level of 520 km above the sun's photosphere show displacements at the center of the sun in agreement with those given by general relativity. Below this level, in the region where 99 per cent of solar lines originate, upward currents exist in the sun's atmosphere [which tend to modify the apparent result]. . . . . The effect vanishes at the edge of the sun, so that for these lines the difference, λ at the edge of the sun minus λ for arc in vacuum, is the predicted Einstein displacement. . . . . This investigation confirms . . . . that the cause of the differences at the center of the sun between solar and terrestrial wave-lengths are the slowing up of the atomic clock in the sun according to Einstein's theory of general relativity, and radial velocities of moderate cosmic magnitude and in probable directions, or equivalent conditions whose effects vanish at the edge of the sun.

I have referred in the foregoing paragraphs to most of St. John's outstanding contributions to astrophysics, but there are other papers of importance besides these which may be referred to in the litera-

[8] *Mt. W. Contr.*, No. 348; *Ap. J.*, **67**, 195, 1928.

ture. He also had charge for many years of the Mount Wilson work on the solar rotation, which must be published by other authors. In all his work we find outstanding those qualities of carefulness, accuracy, and sterling honesty which distinguish the real investigator.

Dr. St. John was president of the Commission on Solar Physics, International Astronomical Union, and of the Committee on Relationships between Solar and Terrestrial Phenomena, International Research Council. He was a member of the National Academy of Sciences; American Association for the Advancement of Science; American Astronomical Society; American Physical Society; American Mathematical Society; History of Science Society; Astronomical Society of the Pacific; Société Astronomique de France; and an associate of the Royal Astronomical Society.

Dr. Adams writes:

> Socially Dr. St. John was a universal favorite. He was especially fond of club life and served two terms as president of the University Club of Pasadena, a post in which he acquitted himself to the general satisfaction of everyone. A remarkably good portrait of him by Seymour Thomas now hangs on the wall of the club house on North Oakland Avenue. He took an active part in all club functions, had a broad interest in questions of the day, and was apparently never so happy as when planted in the middle of some animated discussion, scientific or otherwise. He was also a member of the Athenaeum and a familiar figure about the city, being, as are so many Pasadenans, very fond of walking. During all these years nothing was permitted to interfere with his work in which his interest and energy never flagged. Long after the age at which most men retire, his enthusiasm carried him on, a scientist whose human side as well commanded the love and honor of all who knew him.

I am indebted to Messrs. Adams, Seares, and Babcock of Mount Wilson Observatory and to Miss Francesca St. John for aid and material for this sketch.

A bibliography of St. John's papers issued before 1923 may be found in Poggendorff's *Biographisch-literarisches Handwörterbuch zur Geschichte der Exacten Wissenschaften*, **4**, p. 706, 1904, and **5**, pp. 1210–1211, 1926. The following collection of recent titles (1923–1933) is by Miss Connor of Mount Wilson Observatory:

"The Atmosphere of the Sun and Relativity" (abstract), *Pub. A.A.S.* (30th meeting, Mt. Wilson), **4**, 84–86, 1923; *Pop. Astr.*, **32**, 23–25, 1924.

(With WALTER S. ADAMS.) "Convection Currents in Stellar Atmospheres," *Ap. J.*, **60**, 43–49, 1924; *Mt. W. Contr.*, No. 279.

(With HAROLD D. BABCOCK.) "Pressure and Circulation in the Reversing-Layer of the Sun's Atmosphere," *Ap. J.*, **60**, 32–42, 1924; *Mt. W. Contr.*, No. 278.

(With WALTER S. ADAMS.) "Convection Currents in Stellar Atmospheres," *Proc. Nat. Acad. Sci.*, **10**, 392–394, 1924; *Mt. W. Comm.*, No. 89.

(With WALTER S. ADAMS and HAROLD D. BABCOCK.) "On Pressure and Convection Currents in the Atmospheres of the Sun and Stars" (abstract), *Pub. A.A.S.* (32d meeting, Dartmouth), **4**, 191, 1924; *Pop. Astr.*, **32**, 621–622, 1924.

(With HAROLD D. BABCOCK.) "Note on the Pressure and Currents in the Sun's Atmosphere," *Proc. Nat. Acad Sci.*, **10**, 389–391, 1924; *Mt. W. Comm.*, No. 88.

(With WALTER S. ADAMS.) "An Attempt To Detect Water-Vapor and Oxygen Lines in the Spectrum of Mars with the Registering Microphotometer" (abstract), *Pub. A.S.P.*, **37**, 158–159, 1925.

"The Red Shift of Solar Lines and Relativity," *Proc. Nat. Acad. Sci.*, **12**, 65–68, 1926; *Mt. W. Comm.*, No. 96.

"Rotation of the Sun" (abstract), *Pub. A.A.S.* (34th meeting, Northfield), **4**, 290–291, 1925; *Pop. Astr.*, **33**, 592–593, 1925.

"Report of the Commission on Solar Physics," *Trans. I.A.U.*, **2**, 27–39, 1925.

(With WALTER S. ADAMS.) "An Attempt To Detect Water-Vapor and Oxygen Lines in the Spectrum of Mars with the Registering Microphotometer," *Ap. J.*, **63**, 133–137, 1926; *Mt. W. Contr.*, No. 307.

"Evidence for the Gravitational Displacement of Lines in the Solar Spectrum Predicted by Einstein's Theory," *Ap. J.*, **67**, 195–239, 1928; *Mt. W. Contr.*, No. 348.

"Revision of Rowland's Preliminary Table of Solar Spectrum Wave-Lengths," *Proc. Nat. Acad. Sci.*, **13**, 678–683, 1927; *Mt. W. Comm.*, No. 101.

"Revision of Rowland's Preliminary Table of Solar Spectrum Wave-Lengths" (read at Reno meeting, A.S.P., 1927); (abstract), *Pub. A.S.P.*, **39**, 256–257, 1927.

"Astrophysical Importance of the Rare-Earth Elements" (read at Berkeley meeting, A. Phys. Soc., 1928); (abstract), *Phys. Rev.*, **31**, 918–919, 1928.

(With CHARLOTTE E. MOORE.) "As to Cadmium in the Sun," *Pub. A.S.P.*, **39**, 314–317, 1927.

"Some Results of the Revision of the Rowland Table" (read at Pomona meeting, A.S.P., 1928); (abstract), *ibid.*, **40**, 270–271, 1928.

"Growth in Our Knowledge of the Sun," *ibid.*, **41**, 133–144, 1929.

"Report of the Commission de Physique Solaire," *Trans. I.A.U.*, **3**, 50–76, 1929.

(With CHARLOTTE E. MOORE.) "On the Presence of the Rare-Earth Elements in the Sun," *Ap. J.*, **68**, 93–108, 1928; *Mt. W. Contr.*, No. 364.

(With CHARLOTTE E. MOORE, LOUISE M. WARE, EDWARD F. ADAMS, and HAROLD D. BABCOCK.) "Revision of Rowland's Preliminary Table of Solar Spectrum Wave-Lengths, with an Extension to the Present Limit of the In-

fra-Red." *Carnegie Inst. Pub.*, No. 396; *Papers of the Mt. W. Obs.*, **3**, xxi+238, 1928.

"Elements Unidentified or Doubtful in the Sun: Suggested Observations," *Ap. J.*, **70**, 160–174, 1929; *Mt. W. Contr.*, No. 385.

"The Unit Character of Multiplets," *Ap. J.*, **70**, pp. 312–318, 1929; *Mt. W. Contr.*, No. 389.

"Excitation Potential in Solar Phenomena," *Ap. J.*, **70**, pp. 319–330, 1929; *Mt. W. Contr.*, No. 390.

"Elements in the Sun," *Proc. Nat. Acad. Sci.*, **15**, 789–793, 1929; *Mt. W. Comm.*, No. 107.

"Elements: Identified and Unidentified in the Sun" (read at Berkeley meeting, A.S.P., 1929); (abstract), *Pub. A.S.P.*, **41**, 267–269, 1929.

"The Einsteinian Displacement of Spectral Lines" (read at Eugene meeting, A.S.P., 1930); (abstract), *Pub. A.S.P.*, **42**, 254, 1930.

"Growth in Our Knowledge of the Sun," *Smithsonian Report for 1930*, pp. 177–189, 1931; reprinted from *Pub. A.S.P.*, **41**, 133–144, 1929.

"Variability of Solar Rotation," *Third Report of Commission Appointed To Further the Study of Solar and Terrestrial Relationships*, pp. 116–122, 1932.

"Observational Basis of General Relativity," *Pub. A.S.P.*, **44**, 277–295, 1932.

"Report of Commission de Physique Solaire," *Trans. I.A.U.*, **4**, 34–57, 1933.

SMITHSONIAN INSTITUTION

July 1935

KINEMATICS AND WORLD-STRUCTURE

By H. P. ROBERTSON

ABSTRACT

The *idealized cosmological problem*, in which the nebulae are considered as particles in homogeneous flow, is analyzed from the standpoint of the operational methodology, allowing the fundamental observers the use only of clocks, theodolites, and light-signals. It is found, as an extension of the Helmholtz-Lie solution of the problem of physical space, that such a space-time necessarily admits the introduction of an *invariant Riemannian metric* of precisely the form and generality of that on which the general relativistic theory of cosmology is based, and in terms of which all given elements can be interpreted in the same way as in the relativistic theory.

INTRODUCTION

The increased knowledge of the structure, distribution, and radial motion of extra-galactic nebulae, which has been amassed during the past decade, has brought about a renewed interest in cosmological speculations. Perhaps the least *ad hoc* of these is that offshoot of Einstein's relativistic theory of gravitation most often referred to by the somewhat misleading designation "theory of the expanding universe." It seeks to develop from the general theory of relativity —which has other more firmly established successes to its credit in explaining phenomena within the solar system and within the galaxy —an idealized universe suitable for the gross description of the observed nebular phenomena. This it accomplishes with the aid of a general *a priori* uniformity postulate suggested by the observations themselves, which may or may not survive the test of further observation; the solution which it offers is by no means unique, nor does it seem possible to predict the extent to which the choice it offers will be restricted by further data.[1]

More recently E. A. Milne has put forward a theory, which is based on purely kinematical considerations, eschewing any and all theories of gravitation, although leaving open the possibility of the subsequent imposition of any gravitational theory consonant with the kinematics.[2] In the present form of this theory Milne attempts

[1] For a unified account of this theory see the writer's report, "Relativistic Cosmology," *Rev. Mod. Phys.*, **5**, 62–90, 1933.

[2] A complete account of the present stage of this theory is to be found in Milne's *Relativity, Gravitation and World-Structure* (Oxford, 1935); to be reviewed in an early issue of this *Journal*.

to go beyond his earlier considerations and to derive, in terms of the purely operational methodology, the most general kinematics suitable for a cosmological theory based on a uniformity postulate of the kind suggested by the observational material. In this idealization individual nebulae are to be replaced by "fundamental particle-observers" A, A′, ... , each equipped with a clock, a theodolite, and apparatus for sending and receiving light-signals—these latter considered as corpuscular impulses in order to avoid an indeterminacy foreign to the problem. Briefly stated, the operational viewpoint restricts the observations of each of these fundamental observers to such as can be made on events on his own world-line with the aid of these instruments. The uniformity postulate, which Milne fittingly calls "the cosmological principle," asserts that the description of the whole system, as given by A in terms of his immediate measurements, is to be identical with the description given by any other fundamental observer A′ in terms of his own measurements; as such it is equivalent to that uniformity requirement on which the general relativistic theory is based (although this is denied by Milne). Actually Milne imposes the further restriction that the world-lines of all fundamental observers concurred at a given event O ("creation"), which is then taken as the zero of all their clocks; although (as Milne himself recognizes) this is not a necessary condition, we shall adopt it temporarily in order to avoid circumlocution, and indicate in due time what change will result from its surrender.

Now in spite of the apparent similarity of initial viewpoint between this theory and the kinematical aspects of the general relativistic theory, their conclusions seem quite at variance. The relativistic theory finds that the cosmological speculations may, on assigning appropriate co-ordinates τ, η^a $(a=1, 2, 3)$, to each event E, be based on any Riemannian map whose invariant metric ds^2 is of the form

$$ds^2=d\tau^2-\frac{R^2(\tau)}{c^2}\,du^2\,, \tag{0.1}$$

where $R(\tau)/c$ is an arbitrary function of τ and

$$du^2=h_{\alpha\beta}(\eta^\gamma)d\eta^\alpha d\eta^\beta \tag{0.2}$$

defines any three-space of constant Riemannian curvature k (which may, without loss of generality, be restricted to the values $k=-1$, 0, $+1$). We thus have here a wide range of possibilities, corresponding to the three possible values of k and the choice of the arbitrary function $R(\tau)$—although the condition adopted temporarily above requires that $R(0)=0$. The world-lines of the fundamental observers are given by $\eta^\alpha=\text{const.}$, and the relations between the co-ordinate (τ, η^α) of an event E, as obtained by different observers, are given by transformations of a six-parameter continuous group G_6 on the η^α which leave the auxiliary line-element (0.2) invariant in form as well as in fact, the value of τ being the same for all observers. This Riemannian map has, among others, the following properties of particular interest in the present investigation:

a) The world-line of each fundamental observer is a geodesic of the metric ds^2, and his clock-time along it is measured by this metric.

b) The world-line of every light-signal is a null-geodesic of the metric ds^2.

We shall on occasion have need of a more specific form for the auxiliary metric du^2; we may then, without serious limitation of the astronomical applicability, choose co-ordinates $\eta^\alpha=(\eta, \theta, \varphi)$ in terms of which

$$du^2=d\eta^2+\sigma^2(\eta)[d\theta^2+\sin^2\theta d\varphi^2]\,, \tag{0.3}$$

where

$$\sigma(\eta)=\left\{\begin{array}{l}\sinh\eta,\ (0\leq\eta),\\ \eta,\ (0\leq\eta),\\ \sin\eta,\ \left(0\leq\eta\leq\dfrac{\pi}{2}\right),\end{array}\right\}\quad\text{for } k=\left\{\begin{array}{r}-1\\ 0\\ +1\end{array}\right\} \tag{0.4}$$

In each case $0\leq\theta\leq\pi$, $0\leq\varphi<2\pi$.

Milne, on the other hand, obtains a solution of the full three-dimensional problem (i.e., with three spatial dimensions) only for the case interpreted by him to mean that the fundamental observers are relatively unaccelerated. His results may be expressed in terms of the Minkowski map, whose metric

$$dS^2=dT^2-\frac{dX^2+dY^2+dZ^2}{c^2} \tag{0.5}$$

satisfies the conditions (*a*) and (*b*) above. The relations between privileged observers are here given by that six-parameter group L_6 of Lorentz transformations which have the unique event O as fixed point and which preserve the direction of time. We remark in passing that the quantity

$$\mathrm{T}^2 = T^2 - \frac{X^2+Y^2+Z^2}{c^2} \tag{0.6}$$

is invariant under this group, and that on performing the transformation

$$\left.\begin{aligned} T &= \mathrm{T}\cosh\eta\,, \qquad X = c\mathrm{T}\sinh\eta\sin\theta\cos\varphi\,, \\ Y &= c\mathrm{T}\sinh\eta\sin\theta\sin\varphi\,, \qquad Z = c\mathrm{T}\sinh\eta\cos\theta \end{aligned}\right\} \tag{0.7}$$

the map (0.5) is seen to be contained in (0.1) as the special case $R(\mathrm{T}) = c\mathrm{T}$, $k = -1$. Only on restricting himself to "collinear" events and observers does Milne's attack on the problem of relatively accelerated observers yield some prospect of success; in this case he finds that their mutual relations must be given by transformations of the form

$$T' - \frac{X'}{c} = p\left(T - \frac{X}{c}\right), \qquad T' + \frac{X'}{c} = p^{-1}\left(T + \frac{X}{c}\right), \tag{0.8}$$

where p^{-1} is the inverse of the arbitrary function p. A discrete linear system of fundamental observers satisfying the cosmological principle is in fact obtained on requiring that the transformation between each two consecutive observers of the set be given by the same functions p, p^{-1}, but Milne does not extend this result to the case in which there exists a one-parameter family of such collinear observers—the only one which could offer a starting-point for the solution of the full three-dimensional problem of accelerated observers.[3]

We propose here to analyze the general problem *ab initio*, using the operational methodology throughout and avoiding what Milne chooses to call the "conceptual terms" of the general theory of relativity. We shall be led to the conclusion that, although none of the postulates characteristic of this latter theory are introduced, any

[3] Cf. *ibid.*, p. 357, n. 8.

such attack necessarily leads to the existence of a quadratic differential metric of precisely the form and generality of (0.1), and which furthermore satisfies the conditions (*a*) and (*b*) imposed in general relativity.

1. ONE-DIMENSIONAL KINEMATICS

We begin, then, with an analysis and a completion of Milne's solution of the problem of collinear observers and events. We adopt Milne's tacit assumptions that if two events are connected by a light-path, that light-path is unique, and that the contracting and opening light-cones with vertices at any event other than O each cut the world-line of each fundamental observer in one, and only one, point.[4] A light-signal sent out by a given observer A at time t_1 by his clock is received by a second observer A′ at time t_1' by his, where t_1' is assumed to be a continuous function of t_1. The locus of these light-paths for all $t_1 \geq 0$, extended backward from A as far as possible and forward from A′, is a certain surface which we denote by AA′. Now if the cosmological principle is to be satisfied, this surface AA′ must coincide with the locus A′A of light-paths leaving A′ at times t_2' and arriving at A at times t_2, as otherwise there would exist a one-parameter family of light-paths from any event on the world-line of A to a corresponding event on that of A′—as may be seen by considering the situation as observed by some appropriate intermediate observer A″. Similarly, if any point (other than the common event O) on the world-line of another fundamental observer A″ is on the surface AA′, then his entire world-line is also, and the three observers A, A′, A″ are said to be "collinear."

Consider now the traces on AA′ of all light-cones with vertices at all events E on AA′ (and contained within the light-cone opening out from O, which is in fact the entire universe in Milne's interpretation); these traces, which are possible light-paths, fall into two families of non-intersecting lines such that through each event E there passes one, and only one, line of each family. Let each line

[4] At least within a sufficiently large domain containing the events in question. We concern ourselves mainly with differential geometrical properties, supplemented, as occasion demands, with remarks on the situation arising in case any of the spaces met later are or may be finite in extent.

of one of these families, say that representing light advancing in the direction $A \to A'$, be assigned the parameter t_1 by A, where, as above, t_1 is the time, as measured by his clock, at which the path in question cuts his world-line, and let him similarly assign the parameter t_2 to those of the other family. A′ will then assign to any two such lines of parameters t_1, t_2 (as given by A) *his* clock-times t'_1, t'_2, respectively, at which they cut his world-line. A may now assign the *co-ordinates* (t_1, t_2) to any event E, where t_1 and t_2 are the parameters of the two lines of the first and second families, respectively, which pass through E; similarly A′ will assign to the same event the co-ordinates (t'_1, t'_2) obtained from his own clock readings, as illustrated in Figure 1. In terms of these co-ordinates, the unique event O is (0, 0), the world-line of A is given by the line $t_1 = t_2$, that of A′ by $t'_1 = t'_2$, the light-paths of the first family by $t_1 =$ const. or $t'_1 =$ const., and those of the second family by $t_2 =$ const. or $t'_2 =$ const.; all light-paths on AA′ are thus given in terms of differentials by the vanishing of either of the quadratic differential forms $dt_1 dt_2$ or $dt'_1 dt'_2$.

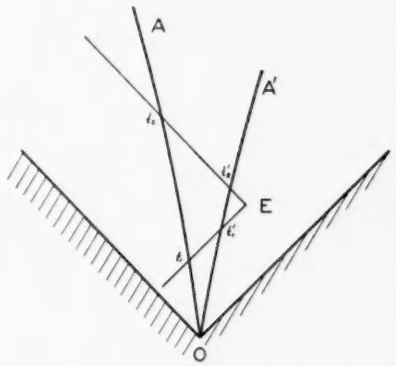

FIG. 1.—*A* assigns to *E* the co-ordinates (t_1, t_2); *A′* the co-ordinates (t'_1, t'_2).

The transformation $t_1, t_2 \to t'_1, t'_2$ from the co-ordinates assigned E by A to those assigned by A′ must clearly consist in a reparameterization of the two families of light-paths, i.e., it must be of the form $t'_1 = p(t_1)$, $t'_2 = q(t_2)$, where p and q are some two (continuous) functions such that $p(0) = 0$, $q(0) = 0$. Further, if the cosmological principle is to be satisfied, this transformation must be such that on interchanging the observers A, A′ and at the same time the two families of light-paths it is left completely unaltered—the interchange of the two families being occasioned by the fact that the parameter t_1 represents a signal advancing in the direction $A \to A'$, and hence must be replaced by a signal t'_2 advancing in the direction $A' \to A$. But this means that t_2, t_1 must be obtained from t'_2, t'_1 by exactly the same

transformation as t_1', t_2' are obtained from t_1, t_2, i.e., $t_2 = p(t_2')$, $t_1 = q(t_1')$. Hence the transformation in question must be of the form

$$t_1' = p(t_1) , \qquad t_2' = p^{-1}(t_2) , \tag{1.1}$$

where p^{-1} is the inverse of the arbitrary function p and $p(0) = 0$, $p^{-1}(0) = 0$; this accomplishes at the same time the standardization of clocks. This result is completely equivalent to that of Milne cited above.

We now complete this solution for the case in which there exists a one-parameter family of possible collinear observers. The transformation (1.1) between any two observers of the family must then be a member of a one-parameter continuous group; hence the transformation p of t_1 must be a member u of a one-parameter continuous group G_1, and t_2 must transform according to its inverse $\breve{u}$, which is also a member of G_1:

$$t_1' = f(t_1; u) , \qquad t_2' = f(t_2; \breve{u}) . \tag{1.2}$$

We may without restriction take $u = 0$ as defining the identical transformation $t_1' = t_1$, $t_2' = t_2$. Now, it is well known that in order for such transformations to form a continuous group, t_1' and t_2', considered as functions of the parameter u, must satisfy a certain set of differential equations;[5] these equations may here, on suitable choice of the parameter u, be taken in the form

$$\frac{dt_1'}{du} = \xi(t_1') , \qquad \frac{dt_2'}{du} = -\xi(t_2') , \tag{1.3}$$

where $\xi(t_1)$ is the generator of the infinitesimal transformation δu of G_1:

$$t_1' = t_1 + \xi(t_1)\delta u \text{ (and } t_2' = t_2 - \xi(t_2)\delta u) . \tag{1.4}$$

The parameter u is hereby determined only to within an arbitrary constant factor C and the generator ξ to within the factor $1/C$; for convenience we limit the choice of this factor by the condition that

[5] Lie's "first fundamental theorem"; cf. S. Lie and F. Engel, *Theorie der Transformationsgruppen*, **3**, Part VI, Leipzig, 1893, to which we shall have occasion to refer in the sequel.

$\xi(t)$ be positive for $t>0$ (or in the range between $t=0$, at which $\xi=0$, and its next zero).

The integration of the fundamental equations (1.3) yields the finite equations (1.2) of the group. Defining

$$F(z)=\int_a^z \frac{dz}{\xi(z)}, \tag{1.5}$$

where a is any suitably chosen lower limit >0 (which never actually enters into the analysis), the finite equations of the group are immediately found to be

$$F(t_1')=F(t_1)+u\,, \qquad F(t_2')=F(t_2)-u\,. \tag{1.6}$$

Each observer A may now characterize each other privileged observer A′ in the set by a fixed value of the continuous parameter u, i.e., by that value of the parameter which defines the transformation sending his measurements into those of A′; he of course assigns to himself the parameter $u=0$, defining the identity. The solution obtained by Milne for a discrete set of such observers is contained herein as the infinite cyclic subgroup $u=n\kappa$, $n=0, \pm 1, \pm 2, \ldots$, where κ is a fixed constant (which may, on suitable normalization, be taken as unity).[6] Note that the group (1.6) admits as its only fundamental invariant $F(t_1)+F(t_2)$; for convenience we choose instead of this quantity itself that solution τ of the equation

$$2F(\tau)=F(t_1)+F(t_2) \tag{1.7}$$

which reduces to $\tau=t_1=t_2$ for events on the world-line of A. All first-order differential invariants are expressible in terms of τ, $dF(t_1)=dt_1/\xi(t_1)$, and $dF(t_2)=dt_2/\xi(t_2)$.

These formal considerations must now be supplemented by an

[6] We have here tacitly assumed, with Milne, that there exists an infinite set of distinct observers, i.e., that they may be thought of as equidistant points along an open straight. If, however, there are but a finite number N of distinct observers, as in the case of N points at equal distances on the circumference of a circle, this subgroup is a finite group of order N, and u must be taken modulo $N\kappa$. From the standpoint of the continuous group, this situation arises if there exists a value of u other than zero which also yields the identity; the significance of this alternative topology for the continuous group will become apparent in the following discussion.

appeal to the empirical in order to determine the hitherto arbitrary function $\xi(t)$ from the given motions. To accomplish this we fix attention on two particular observers A, A′, and complete the normalization of u and $\xi(t)$ by assigning to the transformation A→A′ whichever of the parameter values $u=\pm 1$ leads to positive ξ, as agreed above. A signal sent out by A′ at time $t_1'=t_2'$ is received by A at the time t_2 defined by the equation $F(t_2)=F(t_2')\pm 1$ obtained from (1.6). We may now suppose that A′ informs A of the time t_2' at which the signal was sent, so that t_2 is in principle an empirically determinable function of t_2'; the foregoing equation then determines implicitly the function $F(t)$ and hence $\xi(t)=1/F'(t)$. We come much closer to actual practice, however, by relaxing our too stringent hypothesis concerning the nature of light-signals, and allowing A to determine the "Doppler shift-ratio" $s(t_2)=dt_2/dt_2'=\nu/\nu+\Delta\nu$ directly with the aid of a spectroscope;[7] for that observer A′ characterized by general u this Doppler shift-ratio is given by

$$s(t_2)=\frac{dt_2}{dt_2'}=\frac{\xi(t_2)}{\xi(t_2')}\,, \tag{1.8}$$

where $F(t_2)=F(t_2')+u$. Note that for the infinitesimal transformation (1.4) the foregoing formula becomes $s=1+\xi'(t_2)\delta u$ and would be interpreted by an observer using the traditional formulae for Doppler effect[8] as due to a radial velocity

$$v=c\xi'(t_2)\delta u\,. \tag{1.9}$$

Finally, we may say that, in principle at least, the whole development, including the determination of the generator ξ, is expressible in terms of the operational methodology, although we have, of course, not hesitated to use such purely mathematical tools as seemed most appropriate.

We now ask whether in this general collinear case we can map space-time with the aid of an invariant Riemannian metric ds^2, satisfying, if possible, the conditions (*a*) and (*b*) discussed in the introduction above; this is what is accomplished for the unaccelerated case $\xi(t)=t$ by the Minkowski map. Condition (*b*) requires

[7] Cf. Milne, *op. cit.*, p. 35. [8] Cf. *ibid.*, p. 37.

that such a metric be of the form $f(t_1, t_2)dt_1dt_2$,[9] and the invariance requirement that it be reducible to the form $f(\tau)dF(t_1)dF(t_2)$. For an interval on the world-line of A ($t_1=t_2=\tau$), this latter expression becomes $f(\tau)[d\tau/\xi(\tau)]^2$, and if (a) is to be satisfied, it must be exactly $(d\tau)^2$, i.e., we must have $f(\tau)\equiv[\xi(\tau)]^2$. Hence a metric having the required properties must, if it exists, be of the form

$$ds^2=\frac{\xi^2(\tau)}{\xi(t_1)\xi(t_2)}\,dt_1dt_2\,, \tag{1.10}$$

i.e., it is uniquely determined by the function ξ characterizing the given motions. That the world-line of A, and therefore of any fundamental observer, is a geodesic of this metric is apparent from the symmetry of the latter in t_1 and t_2 (or from the alternative form [1.12] derived below); hence both conditions (a) and (b) are in fact satisfied.

Milne has defined two quantities, the epoch T and the distance X of any event $E(t_1, t_2)$ relative to A, in terms of the quantities t_1, t_2, which are the results of immediate judgments by A;[10] we propose

[9] Thus, in the first instance the fact that the metric is to be quadratic is but an expression of the existence of exactly two families of light-paths; in general, as has been remarked by H. Bateman and by R. Courant in discussing the foundations of the general theory of relativity, a quartic or higher-order metric would imply double refraction or worse (or the introduction of imaginary elements foreign to the problem). This argument, arising from the consideration of light-paths alone, is but supplementary to those adduced by Helmholtz and Lie for the case in which space is considered as given a priori and by Weyl for the case in which it is considered as contingent (cf. references in sec. 2 below); the present investigations are based on an interfusion of these two standpoints. Altogether these arguments should allay the querulous skepticism expressed by Milne (*op. cit.*), p. 342.

[10] *Ibid.*, p. 29; explicitly, $T=(t_2+t_1)/2$, $X=c(t_2-t_1)/2$. Note that our τ satisfies all Milne's requirements for an "epoch" except the quite inappropriate one that on adding a constant to the graduations of the clock carried by A, the epoch be increased by the same amount—inappropriate, because we are here dealing with a situation in which, as previously agreed by Milne (p. 26), the unique event O offers a natural origin for the measurement of time. Note also that for the case of unaccelerated motion our epoch $\tau=(t_1\,t_2)^{\frac{1}{2}}$, i.e., we have merely used the *geometrical* in place of Milne's *arithmetical* mean—surely no great deviation from his methodology! It is hoped that these remarks will serve to dispel the prejudice against "cosmic" time τ, as opposed to "the time [T] used in timing a race, stating athletic records, or arranging a railway time-table" (*ibid.*, p. 44)—incidentally, over here the officials at such experiments usually carry their own watches!

similarly to define two quantities τ, η, which will also serve as the co-ordinates of any event E, but we shall be more noncommittal in naming the second. First, the epoch τ of $E(t_1, t_2)$ is defined by (1.7); it does coincide with the clock time of that fundamental observer A′ whose world-line passes through E, but A has no need of calling upon A′ to determine it. Second, we use in place of distance the u-interval $|\eta|$ between A and the event E in question; η itself is the value of the parameter u of that transformation (1.6) which sends the measurements obtained by A into those obtained by that unique observer A′ whose world-line contains the event E, and is accordingly defined in terms of t_1 and t_2 by the equation

$$2\eta = F(t_2) - F(t_1) \tag{1.11}$$

obtained from (1.6) upon setting $t_1' = t_2' = \tau$ and subtracting. In terms of these co-ordinates (τ, η) the invariant metric (1.10) of the space-time map is readily found to be

$$ds^2 = d\tau^2 - \xi^2(\tau) d\eta^2 , \tag{1.12}$$

and upon setting $\xi(\tau) = R(\tau)/c$, this is seen to be identical with the "one-dimensional" form of the line-element (0.1) upon which the relativistic theory of cosmology is based. We have precisely as much or as little right to assert that its measurement $R(\tau)\eta$ of the "distance" between "simultaneous" events coincides with that obtained with rigid rods as Milne has to identify his co-ordinate X with such measurements in the unaccelerated case. Note that in case we do so, the admittedly conventional assignment (1.9) of a velocity v to the observed Doppler effect in light from that fundamental particle of constant u-interval η would, understandably enough, be consistent with this interpretation only in the first approximation. We choose, however, not to introduce this additional hypothesis or "law," in order to retain the kinematical and general character of the analysis and to avoid premature resort to the empirical.

2. THREE-DIMENSIONAL KINEMATICS

In order to extend these results to the three-dimensional case, consider any pair B, B′ of fundamental observers in the given three-

parameter family, and let BB′ denote the surface containing all events and observers collinear with them. The cosmological principle requires that all that can be said concerning the family AA′ of collinear observers can also be said of the family BB′; thus, in particular, the relations between the various observers in BB′ are given by the same transformation group (1.2), and its infinitesimal generator is therefore the same function $\xi(t)$ to within a constant factor $1/C$. We now normalize the parameter u for this new family by requiring that this factor C be unity; there then exists a unique non-negative u-interval $|u|$ between each pair B, B′ of fundamental observers, which can be determined operationally as described in section 1 above and which vanishes only if B and B′ coincide. Hence there exists a three-dimensional positive-definite metric "u-space" of observers (not to be confused with "physical space") which, in virtue of the cosmological principle, must be homogeneous and isotropic. Now the fundamental Helmholtz-Lie investigations concerning such a space show that *this u-space must admit a positive-definite quadratic metric du^2 of constant Riemannian curvature.*[11] The curvature k of this three-space may, without loss of generality, be restricted to one of the three distinct possibilities $k=-1, 0, +1$ by renormalizing the parameter u (in case $k\neq 0$) with the aid of the constant factor $C=|k|^{\frac{1}{2}}$. On introducing co-ordinates $\eta^\alpha(\alpha=1, 2, 3)$, the metric du^2 of this auxiliary u-space of observers assumes the form (0.2), and each event E in the complete space-time map may be assigned the co-ordinates $(\tau, \eta^1, \eta^2, \eta^3)$. Furthermore, the considerations at the end of the last section and the beginning of this show that we may associate with each pair of "neighboring" events (τ, η^α), $(\tau+d\tau, \eta^\alpha+d\eta^\alpha)$ the invariant interval

$$ds^2=d\tau^2-\xi^2(\tau)du^2\ ; \qquad (2.1)$$

[11] Lie-Engel, *op. cit.*, 3, Part V. For a more direct account of this remarkable theorem (and for further more general results on the uniqueness of the quadratic form in physical geometry) see H. Weyl's Barcelona lectures, *Mathematische Analyse des Raumproblems* (Berlin, 1923). The group of congruences in terms of which the uniformity is expressed (Weyl, *op. cit.*, p. 30) consists here of the transformations from the measurements of any observer A, with arbitrarily oriented theodolite, to those of any other such observer A′, and is described in more detail below.

hence our kinematical space-time map admits an invariant metric of the same form and generality as that which is at the basis of the general relativistic treatment of the cosmological problem. The relations between the fundamental observers A ($\eta^\alpha = \text{const.}$) are given by the group G_6 of automorphisms of du^2, and τ is invariant. The world-lines of these observers A are clearly geodesics, and the light-paths are null-geodesics, as shown below; hence the conditions (*a*) and (*b*) are again fully satisfied.

We return to the operational definition of the co-ordinates η^α and of the group relating the observations of different fundamental observers, the definition of τ having been discussed in section 1 above; with this in mind, we consider first the light-paths in terms of the metric (2.1). A signal sent out by an observer A, whose world-line is given by η_1^α const., and received by another observer A′, η_2^α const., traces out a certain path in the auxiliary u-space between the points η_1^α and η_2^α. Now, by the cosmological principle and the assumption that the light-path between two events is unique, this projection on the u-space must be a geodesic of the auxiliary metric du^2, as otherwise there would exist a one-parameter family of such paths; the considerations of the previous section show that the light-path itself must be a null-line of the metric ds^2. It is readily shown that these two facts together imply that the light-path in the space-time map is necessarily a *null-geodesic* of the metric ds^2,[12] thus proving for the present case the condition (*b*) which is one of the cardinal assumptions of the general theory of relativity. Each observer A may now supplement the epoch τ and the u-interval $\eta^1 = \eta(\geq 0)$ of a distant event E by two further co-ordinates $\eta^2 = \theta$, $\eta^3 = \varphi$ defined as follows. Let him set up his theodolite, once and for all, in any arbitrary way, i.e., by choosing a zenith and a base meridian. He then assigns to E the co-ordinates θ, φ obtained immediately from the altitude $\pi/2 - \theta$ and the azimuth φ of the theodolite when it is set in position to receive light from E. These operationally defined geodesic polar co-ordinates (η, θ, φ) are in fact those employed in the canonical form (0.3) of du^2, and enable A to obtain from immediate judgments alone

[12] This fact may be established by a reversal of the analysis employed in the relativistic theory, for which see Appendix E, p. 87, of the report referred to in footnote 1 above.

the co-ordinates of any event E in his observable universe.[13] Any two of the six-parameter family of co-ordinate systems thus obtained—the choice of an origin (three parameters), of an equatorial plane-element (two), and of a meridian (one)—are related by a transformation of the fundamental group G_6, which is hereby brought into canonical form.

We have thus shown that any space-time satisfying the cosmological principle admits *ipso facto* an invariant quadratic metric ds^2 of the form (2.1) and having the properties (*a*) and (*b*); this result goes beyond the general relativistic treatment, for there a quadratic metric with these latter properties is assumed and is then shown to be necessarily of the form (2.1). No treatment of the cosmological problem, in which the uniformity principle is adopted, can be based on a more general kinematical background than that offered by the relativistic theory; on the other hand, any treatment in which no undue restrictions are imposed must lead to exactly the same general range of possible backgrounds. With regard to this last point, we remark that the assumption that the world-lines of all fundamental observers concurred at $\tau = 0$ (i.e., $\xi(0) = 0$) is clearly not a necessary restriction, and an examination of our analysis shows that it may be dropped, i.e., no restriction of this nature need be placed on $\xi(\tau)$. Further, while we have tacitly assumed "elliptical" rather than "spherical" space in introducing co-ordinates in the auxiliary u-space for the case $k = +1$, we are still free to restore this second possibility —but the decision on observational ground is presumably far beyond our present resources. Finally, we emphasize the fact that the geometry of the three-dimensional u-space is in no sense a matter of convention; whether it is one of (constant) negative, zero, or positive curvature is a purely empirical matter, as is the determination of $\xi(\tau)$.

[13] It follows from eq. (1.6) that light emitted at times τ_0 and reaching A at time τ traverses a u-interval $F(\tau) - F(\tau_0)$; hence A can at time τ have cognizance only of those fundamental particles whose u-interval does not exceed $[F(\tau) - F(0)]/2$, and he can never be aware of an event $E(\tau_0, u)$ whose u-interval exceeds $[F(\infty) - F(\tau_0)]/2$. Unless the integral $F(0)$ is divergent, the first of these anomalies will always be found in the open models $k = -1, 0$, and in the closed ones $k = +1$ for sufficiently small τ; this has led Milne (*op. cit.*, chap. xvii) to reject, on methodological ground, all Friedmann universes, for in them $\xi(\tau) \sim \tau^{\frac{3}{2}}$.

3. INTERPRETATION OF MILNE'S SOLUTION

In our treatment no mention has been made of the "velocity of light"; much less has any question been raised as to its constancy. Indeed, we have not allowed ourselves to identify any property of the metric as giving rise to a "distance" as measured by "rigid rods," as such a procedure would necessitate another, for the present purpose unnecessary, appeal to the empirical. In Milne's attempt to deal with these points is to be found the reason for his failure to obtain the general solution of the problem. Milne has defined co-ordinates X, Y, Z in addition to a time T, has allowed his observers to employ Euclidean geometry with these co-ordinates as a rectangular Cartesian system, and has demanded

c) that the velocity of light c be constant and the same for all observers.

As a result of these assumptions he has of course obtained the well-known result that the differential form (0.5) must be invariant and that the relations between fundamental observers are accordingly given by Lorentz transformations, the linearity of which implies that the observers are in relatively unaccelerated motion; the further requirement that O be fixed restricts the full Lorentz group to that six-parameter subgroup L_6 mentioned in the introduction above, which leaves invariant the quantity T defined by (0.6).[14] He thus obtains the special case $\xi(\tau)=\tau$, $k=-1$ of (2.1) as the only metric satisfying the conditions (*a*) and (*b*) in addition to his condition (*c*), and is led to the conclusion that (*c*) must be altered.[15] Our treatment avoids this *impasse* by making no use of (*c*), but it is also a fact that we could adopt it in certain cases if we are willing to surrender the identification of T with clock time. This procedure, which is based

[14] For the requirement that the vanishing of dS^2 implies the vanishing of dS'^2 means that the transformations between fundamental observers are conformal, and the only conformal transformations in four dimensions are composed of (i) dilatations, (ii) translations, (iii) rotations (here including Lorentz transformations), and (iv) inversions, cf. Lie-Engel, *op. cit.*, **3**, Part IV, or G. Darboux, *Systemes orthogonaux* (Paris, 1910). The physical requirements throw out (i) and (iv), and the requirement that O be fixed leaves only rotations about this event.

[15] *Op. cit.*, p. 51.

on our remark made elsewhere that (0.1) is conformally flat,[16] can of course be expected to lead only to the open models without the introduction of imaginary quantities. It may be of some interest to carry through this interpretation for the case $k=-1$, $F(0)$ divergent, for here the group G_6 becomes exactly that group L_6 on which Milne's considerations are based; if desired, one could then retain for this case, with suitable reinterpretation, much of the formal structure developed in Parts II and III of his book in dealing with accelerated observers.

With this in view, we define a new time-like variable

$$\mathrm{T}=e^{F(\tau)} \tag{3.1}$$

in terms of cosmic time τ, where F is the integral (1.5). The line-element (2.1) for the hyperbolic models $k=-1$ then becomes

$$ds^2=f^2(\mathrm{T})dS^2\,, \tag{3.2}$$

where $f(\mathrm{T})=\xi(\tau)/\mathrm{T}$, upon subjecting the co-ordinates $(\mathrm{T}, \eta, \theta, \varphi)$ to the transformation (0.7) and defining dS^2 as in (0.5). For the cases mentioned above in which $F(0)$ diverges (therefore to $-\infty$), the singular event O becomes the origin of the Cartesian co-ordinates T, X, Y, Z, which is the fixed point O of the Lorentz group L_6. The condition (b) is satisfied by the subsidiary line-element dS^2 as well as by ds^2, for a simple calculation shows that the null-geodesics of one are the null-geodesics of the other; hence Milne's postulate (c) is formally satisfied in terms of the new space-time co-ordinates T, X, Y, Z. However, condition (a) is not fully met by dS^2, for although the world-line of the fundamental observer A is the straight $X=Y=Z=0$ radiating out from O toward the future, T is not a direct measure of clock time along it—except in the case $\xi(\tau)=\tau$ of unaccelerated motion, in which the conformality factor $f(\mathrm{T})$ may be taken as unity and all three conditions (a), (b), and (c) are met simultaneously.

[16] "On the Foundations of Relativistic Cosmology," *Proc. Nat. Acad. Sci.*, **15**, 825, 1929. Cf. also A. G. Walker, *M.N.*, **95**, 263, 1935.

The parabolic case $k=0$ lends itself to similar treatment. In this case it is only necessary to introduce Cartesian co-ordinates X, Y, Z in place of the polar co-ordinates η, θ, φ, for we may then write

$$ds^2=\xi^2(T)dS^2\,, \tag{3.3}$$

where $T=F(\tau)$. The solution is then reduced to one in which the fundamental group G_6 is the full group of Euclidean motions on X, Y, Z. In case $F(0)$ is divergent, the fixed event O recedes to $T=-\infty$, and we are dealing with the whole of the Newtonian universe—the co-ordinate velocity of light being constant merely because we are concerned only with fundamental observers whose mutual distances, as measured by the metric dS^2, are constant. The anomaly stressed by Milne[17] sets in whenever $F(0)$ converges, as in the so-called Einstein–De Sitter universe, for in the parabolic cases "creation" occurred at the finite "time" $T_0=F(0)$ in the past, and in the hyperbolic cases discussed above at "time" $T_0=e^{F(0)}$.

CONCLUSION

We have examined, from the operational standpoint, the problem of determining the most general kinematical background suitable for an idealized universe in which the cosmological principle holds. Allowing the fundamental observers the use only of clocks and theodolites, and granting them the possibility of sending and receiving light-signals, we have shown that for each given mode of motion $\xi(\tau)$ there necessarily exists a quadratic line element (2.1) which is invariant, in form as well as in fact, under transformation from one fundamental observer to another. This metric is determined, apart from topological considerations, to within the sign of the Riemannian curvature of the "space" $\tau=$const., the determination of which requires a second contact with the given and is in no sense conventional. This intrinsically unique metric has the property (*a*) of measuring observers' clock time along each member $\eta^\alpha=$const. of the three-parameter family of fundamental observers' world-lines, which are at the same time geodesics, and the property (*b*) of describing all light-paths as minimal geodesics. More it cannot do

[17] Cf. n. 13 in sec. 2 above.

without additional hypotheses, for we have with this accounted for all given elements—clocks, theodolites, all light-paths, and the world-lines of the given observers. Thus, although this metric map is identical with that to which the kinematical aspects of the general theory of relativity lead on imposing the cosmological principle, it cannot predict the motion of free particles other than the given fundamental particle-observers; it can only restrict their mode of motion by demanding consonance with the uniformity postulate. Any gravitational theory not in conflict with its basic assumptions may be imposed—but the existence of the dimensionless function $\xi(\tau)/\tau$ of τ would appear an insuperable obstacle, even in the idealization here considered, to the development of a purely kinematical-statistical theory of gravitation along the lines proposed by Milne.

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TWO METHODS OF INVESTIGATING THE NATURE OF THE NEBULAR RED-SHIFT*

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ABSTRACT

Two theoretically possible methods of investigating the nature of the nebular red-shift have been known for some time. These two methods depend on the relations connecting either nebular dimensions with observed luminosities or nebular counts with observed luminosities, which might be expected to hold under the alternative assumptions of mutual recession or some other cause for the red-shift. The main purpose of the present article is to study in detail the procedure that must be adopted in applying these methods to actual observations in view of such factors as the indefiniteness of nebular diameters resulting from their gradual decrease in surface brightness from the center outward, the necessity for relating apparent bolometric luminosities to observed photographic magnitudes, and the effect on nebular counts produced by a wide spread in the values of absolute nebular magnitudes.

In addition to this study, preliminary statements are made as to the present status of the observational findings. In the case of the relation between nebular dimensions and luminosities, the observations are such as to confirm our general ideas as to the extra-galactic character of the objects in question, but are not yet sufficient to permit a decision between recessional or other causes for the red-shift. In the case of the relation between nebular counts and luminosities, the observations now available show a rate of increase in counts with distance which seems rather large compared with what would be expected for a homogeneous distribution of nebulae on the basis of either a recessional or a non-recessional theory. Nevertheless, by assuming no appreciable absorption of light on the path from the nebulae and by ascribing a rather high effective temperature to the nebulae, it might be possible to explain the counts on the basis of either a static homogeneous model with some unknown cause for the red-shift or an expanding homogeneous model having a positive spatial curvature which introduces effects that seem unexpectedly large but may not be impossible. This conclusion must be considered as tentative, however, and may need serious modification as a result of further work.

THE NATURE OF THE PROBLEM

1. INTRODUCTION

Light arriving from the extra-galactic nebulae exhibits a shift toward the red in the position of its spectral lines which is approximately proportional to the distance to the emitting nebula. The most obvious explanation of this finding is to regard it as directly correlated with a recessional motion of the nebulae, and this assumption has been commonly adopted in the extensive treatments of nebular motion that have been made with the help of the relativistic theory of gravitation, and also in the more purely kinematical treat-

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ment proposed by Milne. Nevertheless, the possibility that the red-shift may be due to some other cause, connected with the long time or distance involved in the passage of light from nebula to observer, should not be prematurely neglected; and several investigators have indeed suggested such other causes, although without as yet giving an entirely satisfactory detailed account of their mechanism.

Until further evidence is available, both the present writers wish to express an open mind with respect to the ultimately most satisfactory explanation of the nebular red-shift and, in the presentation of purely observational findings, to continue to use the phrase "apparent" velocity of recession. They both incline to the opinion, however, that if the red-shift is not due to recessional motion, its explanation will probably involve some quite new physical principles.

2. PLAN OF TREATMENT

The purpose of the present article is to give detailed consideration to two methods of investigating the nature of the nebular red-shift which, in principle, could be used to distinguish between recessional motion and at least some of the other explanations that might be offered to account for the shift. The first of these methods depends on the relations between nebular dimensions and observed luminosities which would be expected for a homogeneous distribution of nebulae under the alternative assumptions of mutual recession or of some other cause for the red-shift; the second method depends on the relations between nebular counts and observed luminosities which would be expected under those same circumstances. The general theoretical nature of the two methods has been evident for some time, and the main function of the present article is the practical one of investigating the complexities introduced into the treatment of actual observations by such factors as the indefiniteness of nebular diameters connected with their gradual decrease in surface brightness from the center outward, the necessity for relating theoretical luminosities to observed photographic magnitudes, and the effect on nebular counts produced by a wide spread in the actual values of absolute nebular magnitudes.

Part I of the article presents relations which might be expected to hold in case the nebular red-shift is due to recession. In treating the

problem the actual universe is represented by a homogeneous expanding model obeying the relativistic laws of gravitation. This choice of model is recommended by several considerations. The model is governed by the principles of general relativity, which are known to furnish the most adequate description of gravitational phenomena now available. These principles, moreover, are quite definite and provide a sufficient basis for calculating the relations needed for the proposed tests of the nature of the red-shift. The general type of model is familiar, and, by the assignment of suitable values to the different parameters involved in the equations describing the type, a wide variety of special cases becomes available for comparison with observation. Furthermore, it has been shown by the work of Robertson, of Kermack and McRae, and of Milne himself that one of these special cases leads to important observational properties which are approximately the same as would be predicted from Milne's kinematic model—a circumstance which makes it less necessary to give a separate treatment to that model.

Part II of the article presents similar relations for the case that the nebular red-shift is due to causes other than recession. In treating this second problem—prior to any satisfactory detailed theory of such other causes for the red-shift—it is of course not very clear just what model should be taken as a representation of the actual universe. For definiteness we actually use a static Einstein model of the universe, combined with the assumption that the photons emitted by a nebula lose energy on their journey to the observer by some unknown effect, which is linear with distance and which leads to a decrease in frequency without appreciable transverse deflection and, in particular, without any decrease in rate of arrival at the observer. As an objection to the use of this picture it might be urged that the Einstein model is known to be unstable and subject to expansion or contraction if disturbed; and, indeed, the improbability of any kind of mechanics which would permit a stable static distribution of nebulae might also be stressed. Our choice of model is, nevertheless, actually adapted to our present interests since we wish to investigate the possibility that the main cause for the red-shift may not be recession; and we may hence properly use the Einstein static universe as a limiting case of models in which the effect of expansion

or contraction would be negligible compared with that due to the actual (unknown) cause of the red-shift.

Finally, in Part III we make a preliminary statement as to the present status of observational work.

Part I. Treatment Assuming Red-Shift Due to Recession

3. The theoretical formulae

The present section gives certain theoretical formulae which have already been derived as applying to an expanding cosmological model on the basis of relativistic mechanics,[2] while the next section considers the modifications necessary to adapt the formulae for use in connection with actual observations.

For the line element of the model we may take the expression

$$ds^2 = -e^{g(t)}\left(\frac{d\bar{r}^2}{1-\bar{r}^2/R_0^2}+\bar{r}^2 d\theta^2+\bar{r}^2\sin^2\theta d\phi^2\right)+dt^2\,, \qquad (1)$$

where R_0 is a constant parameter related to the pressure and density of material in the model, and the bar over the radial co-ordinate $\bar{r}$ is used to distinguish it from a different radial co-ordinate, r, which is perhaps more often employed but which is less convenient for our immediate purposes. The spacelike co-ordinates $\bar{r}$, θ, and ϕ, in this form of writing the line element, are co-moving in character, so that except for small peculiar velocities the nebulae have no motion relative to these co-ordinates, and the dependence of the model on time is introduced by the appearance of the as yet unspecified function $g(t)$. The timelike co-ordinate t is such as to agree with ordinary measurements of proper time as made by any local observer at rest with respect to $\bar{r}$, θ, and ϕ.

In accordance with this form of line element, the red-shift in the wave-length of light arriving from a distant nebula is given by the formula

$$\frac{\lambda+\delta\lambda}{\lambda} = e^{\frac{1}{2}(g_2-g_1)}\,, \qquad (2)$$

[2] The derivation of these formulae in the same notation as that used here will be found in Tolman's *Relativity, Thermodynamics and Cosmology*, Oxford, 1934. The content of equations (1), (2), and (5) is well known. The original derivation of equations (3) and (4) was given by Tolman, *Proc. Nat. Acad.*, **16**, 511, 1930.

where g_1 is the value of $g(t)$ at the time the light observed leaves the nebula, and g_2 the value of that function at the time of arrival at the observer, both source and observer being regarded as substantially at rest with respect to $\bar{r}$, θ, and ϕ.

For the angular diameters of essentially similar nebulae located at different values of the radial co-ordinate $\bar{r}$, but simultaneously observed at the origin $\bar{r}=0$, we have

$$\delta\theta=\frac{\text{const.}}{\bar{r}}\left(\frac{\lambda+\delta\lambda}{\lambda}\right), \tag{3}$$

where $\delta\lambda/\lambda$ is the observed red-shift for the nebula in question.

For the bolometric luminosity of such similar nebulae, as measured outside the earth's atmosphere in ergs per square centimeter per second, we have

$$l_b=\frac{\text{const.}}{\bar{r}^2}\left(\frac{\lambda}{\lambda+\delta\lambda}\right)^2. \tag{4}$$

And finally for the number of nebulae in a given co-ordinate range $d\bar{r}$, we have

$$dN=\text{const.}\ \frac{\bar{r}^2 d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}}. \tag{5}$$

4. RE-EXPRESSION OF FORMULAE FOR USE WITH OBSERVATIONS

We now turn to the re-expression of the foregoing formulae in forms adapted for application to the discussion of observational findings.

a) *Empirical form for the function* $g(t)$.—Actual observations on the nebulae have not yet reached a stage which will permit a decision between the different theoretically possible forms for the function $g(t)$ that have so far been investigated. Enough progress has been made, however, to show that this function should be taken as approximately linear in t over a long time-interval in the neighborhood of the present. Hence we shall express the function in the form of a power series,

$$g(t)=2(kt+lt^2+mt^3+\cdots\cdots), \tag{6}$$

developed around the time of observation $t_2 = 0$, which is taken for convenience as the starting-point for temporal measurements. The factor 2 in this expression has been introduced to avoid the later appearance of fractions, and the quantities $k, l, m \ldots$, are to be regarded as empirical constants, there being no loss in generality in omitting a constant term in t^0.

b) Red-shift as a function of co-ordinate position.—Substituting (6) into our original expression for the red-shift (2), we may now write

$$z = \frac{\delta\lambda}{\lambda} = e^{-kt_1 - lt_1^2 - mt_1^3 - \cdots - 1}, \tag{7}$$

where t_1 is the time of departure of light from the nebula under consideration, and the symbol z has been introduced as an alternative expression for $\delta\lambda/\lambda$ to simplify the writing of certain later formulae.

This expression relates the red-shift to the time of departure of light from the source t_1, but can readily be changed into forms relating red-shift to the co-ordinate position of the source $\bar{r}$ with the observer at the origin. Setting ds equal to zero in our expression for the line element (1), we can obtain an expression for the co-ordinate velocity of light and thus connect the time of departure t_1 with co-ordinate position $\bar{r}$ by the equation

$$\int_0^{\bar{r}} \frac{d\bar{r}}{(1 - \bar{r}^2/R_0^2)^{\frac{1}{2}}} = \int_{t_1}^{0} e^{-kt - lt^2 - mt^3 - \cdots} dt, \tag{8}$$

which can be shown to lead to a series expression for red-shift in terms of co-ordinate position of the emitting nebula of the form

$$z = \frac{\delta\lambda}{\lambda} = k\bar{r} - l\bar{r}^2 + \left(\frac{k}{6R_0^2} + \tfrac{1}{3}kl + m\right)\bar{r}^3 - \cdots. \tag{9}$$

Observations on red-shift as a function of nebular magnitudes have not yet progressed far enough to determine the coefficients in the foregoing expression beyond the first one, k, which, however, is known with considerable accuracy. On the other hand, since magnitudes can be determined for objects too faint for spectroscopy, it has already become necessary in treating the data on nebular counts to

a given limiting magnitude to extrapolate the expression for red-shift beyond the range over which it has been measured. Since even the signs of l and m are dubious, we shall for the present make the necessary extrapolation by treating these constants as equal to zero. Equation (8) combined with (7) then gives

$$z=\frac{\delta\lambda}{\lambda}=kR_0 \sin^{-1}\frac{\bar{r}}{R_0}=kR_0 \sin^{-1} x \tag{10}$$

as the desired expression for red-shift in terms of co-ordinate position, where we use x as a convenient abbreviation for the ratio $\bar{r}/R_0$.

c) Observed radii for corresponding points on similar nebulae.—The expression for observed angular diameter as a function of co-ordinate position and red-shift given by (3) would be immediately applicable to spherical nebulae having a sharp cut-off in surface brightness at a definite radius. Actual nebulae, however, are found to give photographic images that continue to increase in size with time of exposure, owing mainly to a gradual fading-out of surface brightness as we proceed from the center of the nebula outward. For this reason equation (3) is not directly applicable and is most conveniently replaced by the evident generalization

$$\frac{\rho'}{\rho}=\frac{\bar{r}}{\bar{r}'}\frac{(1+z')}{(1+z)}, \tag{11}$$

where ρ' and ρ are the observed values of the radii for corresponding points on two similar nebulae, located at $\bar{r}'$ and $\bar{r}$ and giving the observed red-shifts z' and z. As corresponding points on the two nebulae we may take points along specified pairs of similar axes at which the surface brightness of the nebulae has dropped to some given fraction of its central value.

d) The relation between co-ordinate position and photographic magnitude.—The relation given by (4) connecting the co-ordinate position and red-shift of similar nebulae with their bolometric luminosities as measured outside the earth's atmosphere is not immediately applicable, since in actual practice the luminosities of nebulae are determined by comparing their photographic magnitudes with those for stars in selected calibrated areas.

To relate photographic magnitudes to bolometric luminosities, we may first define the bolometric magnitude of an object m_b in terms of its bolometric luminosity l_b, and its photographic magnitude m_p in terms of its photographic luminosity l_p, by the equations

$$m_b = 2.5 \log \frac{1}{l_b} + \text{const.} , \tag{12}$$

$$m_p = 2.5 \log \frac{1}{l_p} + \text{const.} , \tag{13}$$

where the additive constants are introduced to permit whatever conventions are desired as to starting-points for the magnitude scales and where the logarithms are to the base 10.

The quantity l_b, called bolometric luminosity, occurring in the first of these equations, has already been defined as the energy flux in ergs per square centimeter per second arriving from the object in question outside the earth's atmosphere, and we may write therefor

$$l_b = \int_0^\infty I(\lambda) d\lambda , \tag{14}$$

where $I(\lambda)d\lambda$ is the energy flux arriving between wave-lengths λ and $\lambda + d\lambda$.

The quantity l_p, called "photographic luminosity," occurring in equation (13), is conceptually less satisfactory than l_b, owing to the complicated nature of photographic processes. Both in the calibration and in the measurement of stellar magnitudes, however, the attempt is made to treat l_p as a definite quantity, which can be obtained by multiplying the energy flux received from the object in question in each wave-length range, $d\lambda$, by a factor proportional to the photographic effectiveness of unit flux at that wave-length and then integrating over all wave-lengths. Hence, if $I(\lambda)$ again refers to the flux outside the earth's atmosphere and $a(\lambda)$ is the fractional transmission through the earth's atmosphere and the instrument in use, we may write

$$l_p = \int_0^\infty I(\lambda) a(\lambda) p(\lambda) d\lambda , \tag{15}$$

where $p(\lambda)$ expresses the photographic effectiveness of unit energy flux at the wave-length λ. We thus obtain, at least formally, a definite expression for l_p, and hence also for m_p, which is defined in terms thereof.[3]

It will be convenient to introduce the symbol p for the ratio of photographic to bolometric luminosity,

$$p=\frac{l_p}{l_b}=\frac{\int_0^\infty I(\lambda)a(\lambda)p(\lambda)d\lambda}{\int_0^\infty I(\lambda)d\lambda}. \tag{16}$$

Subtracting (12) from (13) and introducing (14), (15), and (16), we can then also write

$$\Delta m_{pb}=m_p-m_b=\text{const.}-2.5\log p \tag{17}$$

as an expression for the difference between the photographic and bolometric magnitudes of a celestial object, where the additive constant depends merely on the starting-points selected for the magnitude scales.

We are now ready to consider the relation between co-ordinate position and photographic magnitude. Substituting (16) into (4),

[3] The most serious difficulties connected with the concept of photographic luminosity, as defined by eq. (15), arise from the fact that the values assigned to the luminosities of objects must be determined from the photographic effects which they produce, and such effects depend in general in a complicated, non-linear manner on many factors. The following difficulties may be mentioned: Photographic effects may be estimated in a variety of ways, e.g., from the diameter of a stellar image, the exposure necessary for a threshold image, the density of schraffierkassette images, etc. The extent of such effects is not in general a linear function of intensity of illumination and time of exposure. The relative effects at different wave-lengths depend on the type of plate used, on the total strength of the exposure, and on the method of development. Little is known about the combined effect resulting from the superposition of light of different wave-lengths.

These difficulties are to some extent met as follows: The results of different methods of estimating photographic effects are empirically correlated. Pairs of objects in selected areas of the sky are calibrated as to relative luminosity with the help of neutral screens which make their photographic effects agree; standards of luminosity are thus obtained with which other objects may be compared. Photographic luminosities are regarded as defined with respect to a given type of plate and instrument. The values assigned to $p(\lambda)$ are determined from monochromatic exposures which lead to a density of image comparable with that in astronomical work, when developed in a standard manner.

we obtain for the photographic luminosity of a given nebula as a function of its co-ordinate position $\bar{r}$ and fractional red-shift z the expression

$$l_p = \frac{\text{const.}}{\bar{r}^2}\,(1+z)^{-2}p\ . \tag{18}$$

Combining with the definition for photographic magnitude given by (13) and taking common logarithms, we can write (18) in the form

$$\log \bar{r} = 0.2\{m - 5\log(1+z) + 2.5\log p\} + \text{const.} \tag{19}$$

Furthermore, for convenience, we may choose our scale for the radial co-ordinate in such a way that $\bar{r} = 10$ for an object located at the present time at the usual standard distance of 10 parsecs. As a special case of (19) we may then write

$$\log 10 = 0.2(M + 2.5\log P) + \text{const.}, \tag{20}$$

where M is the absolute magnitude of the nebula in question, the term involving the red-shift has become negligible for this now near-by object, and P is the value given by equation (16) with the spectral distribution which would be shown by the nebula in the absence of red-shift. Subtracting (20) from (19), we then finally obtain, as the desired expression connecting the co-ordinate position of a nebula $\bar{r}$ with its measured photographic magnitude m, absolute photographic magnitude M, and fractional red-shift z,

$$\log \bar{r} = 0.2\{m - 5\log(1+z) - K - M\} + 1\ , \tag{21}$$

where the correction term K is given by

$$K = 2.5\log\frac{P}{p}\ . \tag{22}$$

Two somewhat different methods for evaluating K are available. The first[4] is a semi-empirical method which relates K to the accepted corrections for passing from bolometric to photographic magnitudes for stars of known spectral class; the second[5] is a more computational

[4] Hubble and Humason, *Mt. W. Contr.*, No. 427; *Ap. J.*, **74**, 43, 1931.

[5] W. de Sitter, *Bull. Astron. Inst. Netherlands*, **7**, 205, 1934.

one, involving an integration over the sensitive range of the photographic plate.

To make use of the semi-empirical method, we first note that equations (17) and (22) can be combined to give

$$K = \Delta m_{pb} - \Delta M_{pb} , \tag{23}$$

where ΔM_{pb} is the difference between the photographic and bolometric magnitudes of the nebula which would be found in the absence of any red-shift, and Δm_{pb} is this difference in the presence of the actually existing red-shift. These differences are, of course, immediately related to the unshifted and the shifted spectral distributions in the radiation emitted by the nebula.

From an examination of nebular spectra it proves possible, at least within limits, to identify the unshifted distribution as approximately similar to that from stars of a definite spectral class. Furthermore, making use of the known fairly satisfactory correlation between spectral classes and temperatures, we can regard this unshifted distribution as equivalent to that emitted by a black body at a definite temperature T_0, and take the effect of a superimposed uniform red-shift $\delta\lambda/\lambda = z$ as leading to a shifted distribution such as would be emitted by a source at the lower temperature $T_0(1+z)^{-1}$. Hence it is possible also to identify the shifted distribution as approximately similar to that from stars of a definite later spectral class. We may therefore calculate values of K for any selected initial distribution of radiation and red-shift z by substituting into (23) the accepted values of ΔM_{pb} and Δm_{pb} that have been determined for stars of the spectral classes involved.

The method just outlined may be regarded as semi-empirical, since for stars of any given spectral class we may use the relation

$$\Delta m_{pb} = CI + HI + \Delta m_r , \tag{24}$$

where we can take the color index CI as the actual empirical difference between the photographic and visual magnitudes of such stars, the heat index HI as the empirical difference between their visual and radiometric magnitudes, and the correction to no atmosphere Δm_r as computed from empirical transmission coefficients.

To use the more purely computational method for evaluating the correction K, we may start by combining equations (16) and (22) to give

$$K = 2.5 \log \frac{\int_0^\infty I(\lambda)d\lambda}{\int_0^\infty I_0(\lambda_0)d\lambda_0} \times \frac{\int_0^\infty I_0(\lambda_0)a(\lambda_0)p(\lambda_0)d\lambda_0}{\int_0^\infty I(\lambda)a(\lambda)p(\lambda)d\lambda}, \qquad (25)$$

where $I(\lambda)\,d\lambda$ is the energy flux in the wave-length range λ to $\lambda+d\lambda$ arriving from the actual nebula located at $\bar{r}=\bar{r}$, and $I_0(\lambda_0)d\lambda_0$ is the flux that would be arriving in the range λ_0 to $\lambda_0+d\lambda_0$ from a hypothetical similar nebula at the standard distance where $r=10$.

To obtain actual figures with the help of this expression, we may treat the unshifted distribution $I_0(\lambda_0)$ as corresponding to that from a black body at temperature T_0, and the shifted distribution $I(\lambda)$ as corresponding to the lower temperature $T_0(1+z)^{-1}$, where $\delta\lambda/\lambda=z$ is the fractional red-shift, which, in accordance with the Planck distribution law, gives us

$$K = 2.5 \log \frac{\int_0^\infty \lambda^{-5}\left(e^{\frac{c_2(1+z)}{\lambda T_0}}-1\right)^{-1} d\lambda}{\int_0^\infty \lambda^{-5}\left(e^{\frac{c_2}{\lambda T_0}}-1\right)^{-1} d\lambda} \times \frac{\int_0^\infty \lambda^{-5}\left(e^{\frac{c_2}{\lambda T_0}}-1\right)^{-1} a(\lambda)p(\lambda)d\lambda}{\int_0^\infty \lambda^{-5}\left(e^{\frac{c_2(1+z)}{\lambda T_0}}-1\right)^{-1} a(\lambda)p(\lambda)d\lambda}. \qquad (26)$$

Here the value of the radiation constant c_2 is known, and it has not been necessary to consider the absolute values of I and I_0, owing to their double occurrence in both numerator and denominator. This expression is easily reduced to

$$K = 2.5 \log \frac{\int_0^\infty \lambda^{-5}\left(e^{\frac{c_2}{\lambda T_0}}-1\right)^{-1} a(\lambda)p(\lambda)d\lambda}{(1+z)^4 \int_0^\infty \lambda^{-5}\left(e^{\frac{c_2(1+z)}{\lambda T_0}}-1\right)^{-1} a(\lambda)p(\lambda)d\lambda}. \qquad (27)$$

Referring to (21), we can also, if desired, combine the two corrections to be subtracted from the photographic magnitude m into the form

$$\Delta m = 5 \log (1+z) + K$$

$$= 2.5 \log \frac{\int_0^\infty \lambda^{-5} \left(e^{\frac{c_2}{\lambda T_0}} - 1\right)^{-1} a(\lambda) p(\lambda) d\lambda}{(1+z)^2 \int_0^\infty \lambda^{-5} \left(e^{\frac{c_2(1+z)}{\lambda T_0}} - 1\right)^{-1} a(\lambda) p(\lambda) d\lambda} . \qquad (28)$$

It should be specially noted that this expression differs from the correction to m proposed by de Sitter,[6] which contains the term $(1+z)^3$ instead of $(1+z)^2$. Expression (28), however, would seem to give the proper correction to use in connection with our equation (21), since it has been derived in such a way as to make appropriate allowance, first, for the double effect of nebular recession in reducing both the individual energy and the rate of arrival of photons, and then for the further circumstance that a change in spectral distribution of the energy that does arrive will lead to changes in its photographic effectiveness. It is to be noted that the necessity of allowing for the double effect of recession, both on the individual energy and the rate of arrival of photons, is analogous to the necessity of using the quadratic factor

$$[\sqrt{1 - V^2/c^2}/(1+V/c)]^2 = (1+z)^{-2}$$

in making the ordinary Lorentz transformation for the component of a Poynting vector lying in the direction of the relative velocity V of the two systems of co-ordinates considered.

In concluding this section it may be mentioned that we have already used both the semi-empirical and the computational methods to obtain values of K as a function of the red-shift $\delta\lambda/\lambda = z$, for an assumed range of values for the spectral class and temperature of the nebulae without red-shift, and have found reasonable agreement between the results obtained by the two methods. This now completes the long discussion necessary to replace the simple theoretical relation (4) between co-ordinate position and luminosity, by the prac-

[6] *Loc. cit.*, equations (26), (27), and (28).

tical relation (21) between co-ordinate position and photographic magnitude with its bothersome but necessary correction term K.

e) Allowance for spread in absolute nebular magnitudes.—As the final step in translating the theoretical formulae of section 3 into their practical form, we shall find it desirable to modify formula (5) for the number of nebulae in a given co-ordinate range in such a way as to recognize the actual spread in absolute nebular magnitudes. Since this spread is observationally found[4,7] to be approximately a Gaussian distribution with a certain dispersion σ around a mean absolute magnitude M_0, we can re-write (5) in the form

$$dN = \frac{C}{\sqrt{2\pi}\sigma} e^{-\frac{(M-M_0)^2}{2\sigma^2}} \frac{\bar{r}^2 d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}} dM \ , \tag{29}$$

where C is a constant and dN now becomes the number of nebulae located in the co-ordinate range $\bar{r}$ to $\bar{r}+d\bar{r}$ and having absolute magnitudes between M and $M+dM$.

5. DISTRIBUTION OF LUMINOSITY OVER THE SURFACE OF NEBULAR IMAGES

After the foregoing long preliminary we are now ready to consider the use of our formulae to obtain tests of the nature of the red-shift. Any such test must be expressible in language which eliminates the co-ordinate $\bar{r}$ as the measure of nebular distance, since this is an auxiliary rather than a directly observable quantity. As a first method of eliminating $\bar{r}$, we can combine equations (3) and (4) to give

$$\frac{\delta\theta}{l_b^{\frac{1}{2}}} = \text{const.} \left(\frac{\lambda+\delta\lambda}{\lambda}\right)^2 \tag{30}$$

as a connection between the observable quantities—angular diameter, bolometric luminosity, and red-shift—for the case of intrinsically similar nebulae.[8] For practical purposes, however, this simple formula must be replaced by expressions which allow for the use of photographic methods and for the absence of sharp nebular boundaries.

[7] Hubble and Humason, *Mt. W. Comm.*, No. 116; *Proc. Nat. Acad.*, **20**, 264, 1934.

[8] Tolman, *Proc. Nat. Acad.*, **16**, 511, 1930.

Consider two elliptical nebulae, located at $\bar{r}$ and $\bar{r}'$, but similar in intrinsic character and so oriented as to present similar aspects to an observer at the origin, and let i and i' be the intensities of photographic illumination for corresponding points on their photographic images. Since the total photographic luminosity of an object is proportional to the integrated intensity of illumination over its image, and since for elliptical nebulae the character of the spectral distribution is approximately independent of position on the luminous surface, we can put

$$\frac{i dS}{i' dS'} = \frac{l_p}{l'_p}, \tag{31}$$

where dS and dS' are corresponding elements of area on the two images, and l_p and l'_p are the total photographic luminosities.

In accordance with the expression for the observed radii of corresponding points given by (11), we can write

$$\frac{dS}{dS'} = \frac{\bar{r}'^2}{\bar{r}^2} \frac{(1+z)^2}{(1+z')^2}, \tag{32}$$

while from (18)

$$\frac{l_p}{l'_p} = \frac{\bar{r}'^2 (1+z')^2 p}{\bar{r}^2 (1+z)^2 p'}. \tag{33}$$

Substituting (32) and (33) in (31), we obtain

$$\frac{i}{i'} = \frac{(1+z')^4}{(1+z)^4} \frac{p}{p'} \tag{34}$$

as an expression for the ratio of the intensities of photographic illumination at corresponding points on the two nebular images.

If desired, we can introduce into (34) our present knowledge of the distribution of photographic intensity over the surface of images of elliptical nebulae by using the empirical equation[9]

$$i = \frac{i_0}{\left(\frac{\rho}{a}+1\right)^2}, \tag{35}$$

[9] Hubble, *Mt. W. Contr.*, No. 398; *Ap. J.*, **71**, 231, 1930.

which gives a good expression for the relation between the photographic intensity i at the radius ρ from the center of the elliptical image, in terms of the central intensity i_0 and a parameter a, which, except in the case of circular images, depends on the direction of the radial axis.

Assuming that relation (35) holds for a nebula at $\bar{r}'$ and using the relation between ρ and ρ' given by (11), we can then re-write (34) for a similar nebula at $\bar{r}$ in the form

$$i=\frac{i_0'\,\dfrac{(1+z')^4}{(1+z)^4}\,\dfrac{p}{p'}}{\left\{\dfrac{\rho}{a'}\,\dfrac{\bar{r}(1+z')}{\bar{r}'(1+z)}+1\right\}^2}, \tag{36}$$

so that we may now take

$$i=\frac{i_0}{\left(\dfrac{\rho}{a}+1\right)^2}, \tag{37}$$

with

$$i_0=\frac{\text{const.}}{(1+z)^4}\,p \qquad \text{and} \qquad a=\frac{\text{const.}}{\bar{r}}\,(1+z)\,, \tag{38}$$

as a general expression for the distribution of surface intensity over the images of similar nebulae at different co-ordinate positions. With the aid of (21) and (22), the expressions (38) for i_0 and a can also be written in the form

$$\left.\begin{aligned}\log i_0&=\text{const.}-4\log(1+z)-0.4K\,,\\ \log a&=\text{const.}-0.2m+2\log(1+z)+0.2K\,.\end{aligned}\right\} \tag{39}$$

Since the red-shift z and the photographic magnitude m are directly measurable quantities and the correction term K can be calculated from the spectral distribution of nebular radiation, these results provide a possibility for testing the assumption of nebular recession.

6. NEBULAR COUNTS TO A GIVEN LIMITING MAGNITUDE

We now turn to a second method of eliminating the co-ordinate $\bar{r}$ so as to obtain a test of our model. This is made possible by the pair

of relations which give photographic magnitudes and numbers of nebulae as functions of $\bar{r}$.

In accordance with (29) we have

$$dN=\frac{C}{\sqrt{2\pi\sigma}}\,e^{-\frac{(M-M_0)^2}{2\sigma^2}}\frac{\bar{r}^2d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}}\,dM \tag{40}$$

for the number of nebulae in the co-ordinate range $\bar{r}$ to $\bar{r}+d\bar{r}$, which have an absolute magnitude lying in the range M to $M+dM$. Integrating this expression from $\bar{r}=0$ out to a particular value $\bar{r}=\bar{r}_1$, we obtain for the number of nebulae with the specified absolute magnitude which lie inside of $\bar{r}_1$

$$dN=\frac{C}{3\sqrt{2\pi\sigma}}\,\bar{r}_1^3\left[\frac{3R_0^3}{2\bar{r}_1^3}\left(\sin^{-1}\frac{\bar{r}_1}{R_0}-\frac{\bar{r}_1}{R_0}\sqrt{1-\frac{\bar{r}_1^2}{R_0^2}}\right)\right]e^{-\frac{(M-M_0)^2}{2\sigma^2}}dM\,. \tag{41}$$

For small values of $\bar{r}_1/R_0$ this expression will make the number of nebulae in question nearly proportional to $\bar{r}_1^3$, since the quantity inside the square brackets will then be nearly unity. For increasing values of $\bar{r}_1/R_0$, however, the effects of curvature will become more and more important.

In order to obtain from equation (41) information as to the number of nebulae brighter than a given apparent magnitude m_1, instead of the number out to a given value of the radial co-ordinate $\bar{r}_1$, we may use relation (21) connecting m and $\bar{r}$. For the purposes of our further treatment it will be advantageous to make use of (21) at once to replace the factor $\bar{r}_1^3$ outside the brackets by the corresponding expression in m_1; inside the brackets, however, it will prove convenient for the present to retain $\bar{r}_1$, substituting, for simplicity, our previous notation

$$x=\bar{r}/R_0\,. \tag{42}$$

Carrying this out, we then obtain

$$\left.\begin{aligned}dN=\frac{1000C}{3\sqrt{2\pi\sigma}}\,10^{0.6[m_1-5\log(1+z_1)-K_1-M]}\\ \times\left[\frac{3}{2x_1^3}\left(\sin^{-1}x_1-x_1\sqrt{1-x_1^2}\right)\right]e^{-\frac{(M-M_0)^2}{2\sigma^2}}dM\,,\end{aligned}\right\} \tag{43}$$

which, by a simple transformation, can be written in the form

$$dN=\frac{1000C}{3\sqrt{2\pi}\sigma}\,10^{-0.6(M_0-0.6909\sigma^2)}10^{0.6m_1}10^{-0.6[5\log(1+z_1)+K_1]}\times\left[\frac{3}{2x_1^3}\left(\sin^{-1}x_1-x_1\sqrt{1-x_1^2}\right)\right]e^{-\frac{(M-\overline{M})^2}{2\sigma^2}}dM\,, \tag{44}$$

where

$$\overline{M}=(M_0-1.3818\sigma^2)\,. \tag{45}$$

This gives us an expression for the number of nebulae having an absolute magnitude between M and $M+dM$ and an apparent magnitude less than m_1, the quantities z_1, K_1, and x_1 being the red-shift, correction term, and value of $\bar{r}/R_0$ for a nebula of apparent magnitude m_1 and absolute magnitude M.

We may now integrate expression (44) over all possible values of M from minus to plus infinity, holding m_1 constant and letting z_1, K_1, and x_1 assume the values which they have for this value of m and the varying values of M. We thus obtain for the total number of nebulae out to the apparent magnitude m_1

$$N_1=\frac{\text{const.}}{\sqrt{2\pi}\sigma}\,10^{0.6m_1}\int_{-\infty}^{+\infty}10^{-0.6[5\log(1+z_1)+K_1]}\times\left[\frac{3}{2x_1^3}\left(\sin^{-1}x_1-x_1\sqrt{1-x_1^2}\right)\right]e^{-\frac{(M-\overline{M})^2}{2\sigma^2}}dM\,, \tag{46}$$

where the constant is independent of the particular value of m_1 under consideration.

The only practical method of performing the integration indicated in equation (46) is to use arithmetical or graphical quadrature. To carry this out we can prepare a table with the red-shift $z=\delta\lambda/\lambda$ taken as a convenient choice for the independent variable.

With the help of equation (10),

$$z=\frac{\delta\lambda}{\lambda}=kR_0\sin^{-1}\frac{\bar{r}}{R_0}=kR_0\sin^{-1}x\,, \tag{47}$$

we can first compute the values of $\bar{r}$ and x which correspond to the listed values of z. In doing this we take for k the relatively well-determined observational value of Hubble and Humason; on the other

hand, in our present state of considerable ignorance as to the density of matter in the universe, we can best regard R_0 as an adjustable parameter, and test a variety of choices, including $R_0 = \infty$, which would correspond to no appreciable effects of curvature on nebular counts.

With the help of the methods of section 4*d*, we can next obtain values for the correction term K as a function of the listed values of z which may be represented by

$$z \longleftrightarrow K . \tag{48}$$

In doing this we must take cognizance of the fact that the isolated, non-cluster nebulae to which our counts will apply have the average character of intermediate-type spirals. From a knowledge of the spectra of such objects we can then make an appropriate choice of a reasonable black-body temperature, or spectral class, or combination of spectral classes, by which to represent the unshifted distribution of radiation.

Finally, with the help of equation (21),

$$\log \bar{r} = 0.2\{m - 5 \log (1+z) - K - M\} + 1 , \tag{49}$$

we can introduce a column giving the values of the absolute magnitude M that correspond to the values of z, $\bar{r}$, x, and K already listed, and to the particular choice of $m = m_1$ which we are studying.

In this way we can determine all the quantities which occur in the integral in (46) as functions of M and thus make the desired quadrature.

By the application of such methods of computation we have found that on account of the fairly sharp maximum exhibited by the exponential factor $e^{-\frac{(M-\overline{M})^2}{2\sigma^2}}$, the integral in (46) can often be satisfactorily approximated by assigning to the rest of the integrand the value that it has at the maximum point where $M = \overline{M}$. We then obtain

$$N_1 = \text{const. } 10^{0.6[m_1 - 5 \log (1+z_1) - K_1]} \left[\frac{3}{2x_1^3} \left(\sin^{-1} x_1 - x_1 \sqrt{1 - x_1^2} \right) \right] , \tag{50}$$

or, by taking logarithms,

$$\log N_1 = 0.6\left\{ m_1 - 5\log(1+z_1) - K_1 + \tfrac{1}{6}\log\left[\frac{3}{2x_1^3}\left(\sin^{-1}x_1 - x_1\sqrt{1-x_1^2}\right)\right]\right\} + \text{const.}, \tag{51}$$

where z_1, K_1, and $x_1 = \bar{r}_1/R_0$ are to be given the values which they would have for a nebula of apparent magnitude m_1 and absolute magnitude $M = \overline{M}$.

In accordance with (45), the value $\overline{M}$ is related to the mean value M_0 for the absolute magnitudes of the nebulae by the equation

$$\overline{M} = M_0 - 1.3818\sigma^2. \tag{52}$$

Using the actual observational value for the dispersion σ, we find $\overline{M}$ to be about 1 mag. brighter than the true mean M_0. This consequence of the presence of dispersion may be regarded as a selection effect, the absolutely brighter nebulae being favored in the counts as compared with the fainter ones, since they reach to greater distances before their apparent magnitudes assume the particular limit m_1 set for the count in question.[7, 10]

Since all the quantities in (51) are empirically determinable, at least as soon as we have chosen a value for the parameter R_0, this equation provides a second possibility for observational tests of the model. It is to be noted, however, in contradistinction to the previous possibility provided by equations (39) for the distribution of surface intensity over the images of similar nebulae, that equation (51) involves the assumption of uniform nebular distribution fully as much as that of nebular recession.

Part II. Treatment Assuming Red-Shift Not Due to Recession

7. The theoretical formulae in the absence of recession

We now turn to a determination of the results to be expected in case the red-shift is due to some cause other than recession. For this purpose it will be sufficient to indicate the modifications in our pre-

[10] See Hubble, *Halley Lecture*, Oxford, 1934.

vious equations arising from the changed point of view. The numbers of the new equations are those of the analogous preceding ones with an accent added.

As already mentioned, the original Einstein static universe is taken as a suitable model for the case now considered in which the red-shift is not due to recession. For the line element we then have

$$ds^2 = -\frac{d\bar{r}^2}{1-\bar{r}^2/R_0^2} - \bar{r}^2 d\theta^2 - \bar{r}^2 \sin^2\theta d\phi^2 + dt^2 . \qquad (1')$$

Since there would be no systematic motion of the nebulae in such a model, we must assume some other unknown cause for the red-shift. In the absence of knowledge as to the mechanism of this cause, we shall take the fractional red-shift as proportional to the proper distance to the nebula, namely,

$$\frac{\delta\lambda}{\lambda} = k\int_0^{\bar{r}} \frac{d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}} . \qquad (2')$$

Somewhat different laws for the red-shift might perhaps seem more plausible, but (2′) provides us with an expression which lies within the limits prescribed by the known data and leads to the same form of relation between red-shift and co-ordinate position as was used in treating the recessional theory.

We assume that the unknown cause for the red-shift acts so as to decrease the energy associated with each photon but not to deflect it appreciably from its geodesic path. For the angular diameters of similar nebulae, located at different values of $\bar{r}$ but simultaneously observed at the origin, we then have

$$\delta\theta = \frac{\text{const.}}{\bar{r}} . \qquad (3')$$

Furthermore, since the new cause for the red-shift is assumed to decrease the energy of photons without affecting their rate of arrival, we may write for the bolometric luminosity of similar nebulae at different co-ordinate positions

$$l_b = \frac{\text{const.}}{\bar{r}^2}\left(\frac{\lambda}{\lambda+\delta\lambda}\right) . \qquad (4')$$

Finally, for the number of nebulae in a given co-ordinate range, we again have

$$dN = \text{const.}\ \frac{\bar{r}^2 d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}}\,. \qquad (5')$$

8. RE-EXPRESSION OF FORMULAE FOR USE WITH OBSERVATIONS IN THE ABSENCE OF RECESSION

The consequences of the foregoing theoretical formulae may now be expressed in forms adapted to the discussion of observational findings, as was done in section 4 for the case of recession. Since the derivations are similar to the earlier ones, we merely state the new formulae which are of special interest.

For the red-shift as a function of co-ordinate position, as before,

$$z = \frac{\delta\lambda}{\lambda} = kR_0 \sin^{-1}\frac{\bar{r}}{R_0} = kR_0 \sin^{-1} x\,. \qquad (10')$$

For the ratio of the observed radii ρ' and ρ for corresponding points on similar nebulae, we have the new expression

$$\frac{\rho'}{\rho} = \frac{\bar{r}}{\bar{r}'}\,. \qquad (11')$$

The new relation between co-ordinate position $\bar{r}$ and photographic magnitude is

$$\log \bar{r} = 0.2\{m - 2.5 \log (1+z) - K - M\} + 1\,, \qquad (21')$$

where K has the same value as before, but the total correction to the photographic magnitude m is smaller than before by the amount $2.5 \log (1+z)$, owing to the fact that there is no longer any recessional action to decrease the rate of arrival of photons.

Finally, for the number of nebulae in the co-ordinate range $d\bar{r}$ having an absolute magnitude in the range dM, we again have the expression

$$dN = \frac{C}{\sqrt{2\pi\sigma}} e^{-\frac{(M-M_0)^2}{2\sigma^2}} \frac{\bar{r}^2 d\bar{r}}{(1-\bar{r}^2/R_0^2)^{\frac{1}{2}}} dM\,, \qquad (29')$$

where C is a constant.

9. DISTRIBUTION OF LUMINOSITY OVER THE SURFACE OF NEBULAR IMAGES IN THE ABSENCE OF RECESSION

We now consider the relations between nebular luminosities and dimensions to be expected in case the red-shift is not due to recession. Combining (3′) and (4′), we obtain

$$\frac{\delta\theta}{l_0^{\frac{1}{2}}}=\text{const.}\left(\frac{\lambda+\delta\lambda}{\lambda}\right)^{\frac{1}{2}}, \tag{30′}$$

as the relation between observed diameters, bolometric luminosities, and red-shift for similar nebulae at different distances. It differs from the previous expression by the entrance of the red-shift term to the power $\frac{1}{2}$ instead of 2. The result in this simple form, however, applies as before to objects which are thought of as having a sharp cut-off in surface brightness at some definite radius.

For actual elliptical nebulae with a surface brightness which decreases gradually from the center outward, we obtain

$$\frac{i}{i'}=\frac{(1+z')\,p}{(1+z)\,p'}\,. \tag{34′}$$

This expression for the ratio of the intensities of photographic illumination at corresponding points on the images of two essentially similar nebulae at different distances from the observer contains the red-shift terms to the power 1 instead of 4 as in the previous result.

Furthermore, introducing the known observational facts as to the distribution of the surface brightness of elliptical nebulae, we again obtain

$$i=\frac{i_0}{\left(\frac{\rho}{a}+1\right)^2} \tag{37′}$$

for the photographic intensity i as a function of radial position ρ, but the parameters i_0 and a for similar nebulae at different distances now satisfy the appreciably altered relations

$$\left.\begin{aligned}\log i_0&=\text{const.}-\log(1+z)-0.4K\,,\\ \log a&=\text{const.}-0.2m+0.5\log(1+z)+0.2K\,.\end{aligned}\right\} \tag{39′}$$

Since all the quantities in (37′) and (39′) are empirically determinable, comparison with the results from (37) and (39) should give a possibility of distinguishing between the two types of explanation for the red-shift.

10. NEBULAR COUNTS TO A GIVEN LIMITING MAGNITUDE IN THE ABSENCE OF RECESSION

The two types of explanation for the red-shift lead also to different expectations as to the dependence of nebular counts on apparent magnitude. In the absence of recession, we should have

$$\log N_1 = 0.6\left\{ m_1 - 2.5\log(1+z_1) - K_1 + \tfrac{1}{6}^{0}\log\left[\frac{3}{2x_1^3}\left(\sin^{-1}x_1 - x_1\sqrt{1-x_1^2}\right)\right]\right\} \qquad (51')$$

for the number of nebulae N_1 brighter than apparent photographic magnitude m_1, where z_1, K_1, and $x_1 = \bar{r}_1/R_0$ are again to be given the values which they would have for a nebula of apparent magnitude m_1 and absolute magnitude $\overline{M} = M_0 - 1.3818\sigma^2$. The expression differs from the previous one by containing the term $2.5\log(1+z_1)$ instead of $5\log(1+z_1)$, thus providing another possibility of distinguishing between the two types of explanation for the red-shift.

PART III. PRESENT STATUS OF OBSERVATIONAL WORK

11. INVESTIGATION OF THE STRUCTURE OF NEBULAR IMAGES

Attention has been given in the Mount Wilson program of observation to both the foregoing methods of attack, and a preliminary statement as to the present status of the observations will not be out of place in view of our natural desire to understand the causes for such a general phenomenon as the nebular red-shift.

Consider first the method provided by the theoretical relations for the distribution of intensity over the surfaces of similar nebular images. These relations were derived for the case of elliptical nebulae in which differences between the spectral distribution of light from the center and that from the periphery of the object could be neglected. Since the great nebular clusters are composed largely of elliptical nebulae, all in any one cluster being at approximately the

same distance from us, the photographic study of clusters should provide especially suitable data. The investigation can be made by exposing to clusters at different distances for the same length of time and then using the empirical relations between intensity of photographic illumination and resulting photographic effect for a given exposure time.[9]

On the basis of both theories we should find nebular images in the different clusters with photographic intensity i given as a function of radius ρ by relations of the form

$$i=\frac{i_o}{\left(\frac{\rho}{a}+1\right)^2}. \qquad (37, 37')$$

The two theories would lead to different expectations, however, as to the dependence on distance of the parameters i_o and a, for a given kind of nebula, as we go to photographs of more and more distant clusters.

We may first inquire whether this difference between the two theories could be detected by measuring the over-all dimensions to which the nebular images in the different clusters build up in their prescribed period of exposure. Denoting by i_{th} the threshold intensity which is just sufficient to give a detectable effect in the prescribed period, we can solve (37, 37′) for the peripheral radius ρ_{th} in the form

$$\rho_{th}=a\left(\frac{i_o^{\frac{1}{2}}}{i_{th}^{\frac{1}{2}}}-1\right). \qquad (53)$$

And since the central intensity i_o will be very large compared with the threshold intensity i_{th}, (53) can be written in the form

$$\log \rho_{th}=\log a+0.5 \log i_o-0.5 \log i_{th}. \qquad (54)$$

For different distances, however, the threshold intensity i_{th} will be the same, while for a and i_o we can substitute expressions (39) and (39′), which relate these parameters to distance on the basis of the two theories. We thus obtain for both theories exactly the same form of expression

$$\log \rho_{th}=\text{const.}-0.2m. \qquad (55)$$

In accordance with this expression the over-all dimensions of the images would decrease according to the same law for both theories as we go to fainter and hence larger magnitudes. This necessary conclusion is unfortunate since a determination of the over-all dimensions of nebular images with the help of microphotometric analysis is the most practical measurement of their structure that we can make.

The preliminary observations on cluster nebulae thus far made at Mount Wilson have been in agreement with equation (55). Even within any given cluster the dimensions and magnitudes of the nebulae are found to be connected by an approximately linear relation of the form (55), presumably on account of some dependence of total luminosity on dimensions; and then on passing from one cluster to a more distant one the same relation continues to hold without appreciable change in the value of the constant. This check of a relation which would be expected to hold on both theories can serve to give an increased feeling of confidence in our general picture of nebulae and nebular clusters as really being extra-galactic objects at different distances from us, but cannot distinguish between the two suggested explanations of the red-shift.

To obtain criteria for distinguishing the two types of explanation, with the help of our analysis of the surface distribution of nebular luminosity, a more detailed study of the parameters i_0 and a for the nebulae in different clusters would be necessary. Expressions (39) and (39′) for these quantities in the case of essentially similar nebulae at different distances can be re-written in the forms

$$\left.\begin{aligned} i_0(1+z)^4 10^{0.4K} &= \text{const.} \\ \frac{a 10^{0.2m}}{(1+z)^2 10^{0.2K}} &= \text{const.} \end{aligned}\right\} \text{(with recession)}$$

and

$$\left.\begin{aligned} i_0(1+z) 10^{0.4K} &= \text{const.} \\ \frac{a 10^{0.2m}}{(1+z)^{\frac{1}{2}} 10^{0.2K}} &= \text{const.} \end{aligned}\right\} \text{(without recession)}$$

Since all the quantities in these formulae are measurable, the quite different dependence on the red-shift $z=\delta\lambda/\lambda$ for the two theories

should be detectable, at least in principle, by determining the distribution of the foregoing quantities among the individual nebulae of clusters at different distances.

Unfortunately, this program of possible investigation is made very difficult by the necessity of going to quite distant clusters in order to get sufficiently large red-shifts. The nebular images then become so small compared with microphotometer aperture and size of plate grain that determinations of the central intensity i_0 or of the radius a where the intensity has fallen to one-quarter are not easy. It is to be hoped, nevertheless, that further development of technique with the 100-inch telescope or the use of the 200-inch telescope now under construction will at some time make this method of study applicable.

12. DETERMINATION OF NEBULAR COUNTS

We now turn to the method of investigating the nature of the red-shift provided by the determination of nebular counts to successive limits of apparent magnitudes. Here, in accordance with equations (51) and (51′), the two theories lead to an expression for the number of nebulae N_1 to a given limiting magnitude m_1 of the same form,

$$\log N_1 = 0.6(m_1 - \Delta m_1) + \text{const.}, \tag{56}$$

where, however, the correction Δm_1 to be applied to the apparent magnitude m_1 depends on the type of explanation proposed for the red-shift.

In the absence of any red-shift and with a uniform distribution of nebulae in Euclidean space, the correction Δm_1 would, of course, be zero, and the foregoing expression would reduce to the familiar form

$$\log N_1 = 0.6m_1 + \text{const.} \tag{57}$$

The fact that this simple equation does give an approximate expression for the observational results[11] out to distances where the effects of red-shift and presumable spatial curvature would be unimportant may be taken as agreeing with our general picture of the extra-galactic distribution of nebulae.

Allowing, however, for the actual presence of a red-shift in the

[11] Hubble, *Mt. W. Contr.*, No. 485; *Ap. J.*, **79**, 8, 1934.

light from the nebulae and treating them as uniformly distributed in a space of constant curvature, we have found the correction term to have the form

$$\Delta m_1 = 5 \log (1+z_1) + K_1 - \tfrac{1}{6}^0 \log \left[\frac{3}{2x_1^3} (\sin^{-1} x_1 - x_1 \sqrt{1-x_1^2}) \right] \quad (58)$$

for the case of recession as the cause of the red-shift, and the form

$$\Delta m_1 = 2.5 \log (1+z_1) + K_1 - \tfrac{1}{6}^0 \log \left[\frac{3}{2x_1^3} (\sin^{-1} x_1 - x_1 \sqrt{1-x_1^2}) \right] \quad (58')$$

on the basis of a non-recessional theory of the phenomenon. In both of these expressions the fractional red-shift z_1, the correction term K_1, and the ratio $x_1 = \bar{r}_1/R_0$ are to be taken in accordance with our previous discussion as the values for a nebula having the apparent magnitude m_1 and the absolute magnitude

$$\overline{M} = M_0 - 1.3818\sigma^2 , \quad (59)$$

where M_0 is the mean absolute magnitude for isolated nebulae and σ the dispersion around this mean.

In order to make tests of the differences implied by equations (58) and (58′), it is necessary to have a homogeneous body of nebular counts to a series of different limiting magnitudes, which go out far enough to make the effects of red-shift appreciable. For each limiting magnitude the areas of the sky covered must be so selected as to give a representative sample of isolated non-cluster nebulae, and the different limiting magnitudes themselves must be determined by methods having sufficient accuracy and interconsistency. The attainment of such data forms a definite part of the Mount Wilson program of observation.

At present the nearest approach to a satisfactory body of data is provided by published[12] and unpublished[13] nebular surveys made at Mount Wilson with several different exposure times at the 60-inch

[12] Hubble, *Mt. W. Contr.*, No. 485; *Ap. J.*, **79**, 8, 1934, giving surveys to $m = 19.4$ and $m = 20.0$, made with one-hour exposures at the 60-inch and 100-inch reflectors.

[13] Surveys by Hubble to $m = 18.5$ and $m = 21.0$ made with twenty-minute exposures at the 60-inch and two-hour exposures at the 100-inch reflector.

and 100-inch reflectors, combined with the survey of Mayall[14] at Mount Hamilton based on one-hour exposures with the 36-inch reflector. Each of these surveys provides a set of plates, on which the numbers of nebulae have been counted down to a definite threshold size and density of image, so chosen that the distinction between stars and nebulae is still sufficiently reliable, together with an independent determination of the apparent magnitudes of the nebulae giving these threshold images. In this way values have been obtained for the average number of isolated nebulae per square degree brighter than a series of limiting magnitudes estimated as $m = 18.5$, 19.0, 19.4, 20.0, and 21.0.

The time and effort needed for obtaining such a set of surveys, together with the necessity for unique telescopic facilities which must also be used for other tasks as well, make it desirable to employ the present data to the fullest possible extent. This makes it necessary to draw tentative conclusions which are affected by difficulties and uncertainties as to the best analysis of the present data and may require serious modification as further data accumulate. We must now consider the nature of some of these difficulties and uncertainties. We shall take up the different quantities involved in the order in which they occur in the equations to be tested.

Turning first to the nebular counts N_1, it will of course be obvious that the actual process of determining the number of nebular images per plate is difficult and laborious, involving the use of systematic methods of counting and checking, as well as the maintenance throughout each survey of a uniform threshold of identification at a level giving at least an approximate balance between the numbers of stellar and nebular images wrongly identified. In addition, however, in order to secure a homogeneous body of data, the actual count on each plate must be subjected to a number of corrections to make it correspond to some selected set of standard conditions. In practice[11] this means multiplication by a series of factors to obtain a corrected count corresponding to the atmospheric extinction at the zenith, to the galactic obscuration at the galactic pole, to plates of "excellent" quality, to the standard time of exposure selected for the

[14] *Lick Obs. Bull.*, **16**, 177 (No. 458), 1934, giving a survey to $m = 19.0$, made with one-hour exposures at the 36-inch reflector.

survey, and to the photographic definition at the center of the plate. Some uncertainty is undoubtedly introduced by the necessity for such elaborate correction; moreover, this uncertainty must not be minimized since the corrections up to "excellent" quality of plate are so much more important for the most distant as compared with the nearest survey that quite different conclusions would be drawn from a direct use of the uncorrected counts. As a final remark concerning N_1, it may be mentioned that the different plates of a survey, especially of surveys for the less distant nebulae, show a wide spread in the corrected counts. Nevertheless, a study of the character of the dispersion shows that the actual mean counts give values in good agreement with what would be expected for an infinite collection of such survey plates.

Turning next to the determination of the limiting magnitudes m_1 for the different surveys, we meet the usual difficulties in accurate magnitude determinations, and, in addition, a special difficulty connected with the very faint magnitudes to which the surveys must be carried in order to make the desired tests. The most reliable determinations of magnitudes are to be obtained with the help of schraffierkassette images. And for the nearer surveys it is possible with sufficient exposure time to obtain such images for objects that do give threshold images under survey conditions. The more distant objects, however, are too faint to give such images with exposure times that are possible, and methods of extrapolation must be used. These methods, which cannot be discussed here, have, to be sure, led to results of considerable internal consistency. Nevertheless, an error thus introduced which was itself a function of magnitude might be serious.

We must next consider the fractional red-shift $z = \delta\lambda/\lambda$, which is a quantity of particular interest because of its different occurrence in the alternative expressions for the correction to photographic magnitude, Δm_1, given by the two theories. With the help of spectroscopic and photometric measurements made on selected clusters, the relation between red-shift and magnitude has now been determined for red-shifts up to about 0.13. The results show a linear dependence of $\delta\lambda/\lambda$ on the co-ordinate $\bar{r}$, within limits of accuracy such that either of the formulae, (21) or (21′), given by the two theories

may be employed for the calculation of the co-ordinate $\bar{r}$ from the measured magnitude m. Hence, with allowances for the possibility that spatial curvature may have an effect at still larger distances, it seems reasonable, as already suggested, to make the extrapolation to greater distances with the help of equation (10). Nevertheless, the uncertainties involved in the long extrapolation must not be minimized. For the nearest survey, to be sure, the value of $\delta\lambda/\lambda$ is found to be only about 0.08 for a nebula having the apparent magnitude $m = 18.5$ and the absolute magnitude $M = \overline{M}$, which in accordance with our discussion of equation (46) is a suitable "average" nebula to use in calculating the correction term Δm_1. On the other hand, for the most distant survey the value of $\delta\lambda/\lambda$ would be about 0.20 for a nebula having the apparent magnitude $m = 21.0$ and the absolute magnitude $M = \overline{M}$, which lies well beyond the limit of about 0.13 up to which measurements have been made. Furthermore, it is to be emphasized that the precise evaluation of the integral occurring in equation (46) is appreciably dependent on the value of $\delta\lambda/\lambda$ for a range of distances on both sides of that corresponding to m_1 and $\overline{M}$. Hence the necessity of extrapolation introduces uncertainties which are unavoidable at present but may be serious.

We must now turn to difficulties connected with the correction term K_1. The methods for relating this quantity to an assumed spectral type or an assumed black-body temperature for the nebulae involved have already been discussed in section 4*d*. Difficulties arise, however, as to what spectral type or black-body temperature to choose. The isolated nebulae involved in the counts range through all types with a reasonable average around Sb spirals. The greater proportion of the effective radiation from such spirals is known to come from the inner portions of the nebulae and to have a spectrum similar to that from stars of approximately the type dG2, which corresponds to an effective temperature of about $T_0 = 5840°$. There are two complicating factors, nevertheless, which affect the situation in opposite directions. First, there are blue stars of much earlier type in the outer regions of such nebulae which would raise the effective temperature by an unknown but perhaps not very large amount; while, on the other hand, unpublished measures with photoelectric

cells and color filters made by Stebbins and his associates at the foci of the two Mount Wilson reflectors have given color indices for the inner regions of such nebulae which would correspond to effective temperatures perhaps as low as $T_0 = 4860$. Further work, including the use of red-sensitive plates and color screens for the photographic determination of the color indices of distant nebulae, may clarify the situation. In the meantime, however, any statement as to the conclusions to be drawn from nebular counts have to be judged in the light of some accompanying statement as to the spectral type or effective temperature assumed in the calculation of the correction term K_1.

We turn now to the final constant term occurring in equation (56) for the expected number of nebulae to different limiting magnitudes. Here it may be remarked that the constant could be determined, either to give the best fit for the five surveys under consideration or in such a way as to give weight to counts that have been made to numerically smaller magnitudes at other observatories. The assignment of proper weight to such other results is difficult, however, owing to differences in methods of magnitude determination and to the possibility that the number of nebulae in our immediate neighborhood may be affected by a fluctuation from the mean.

To complete the discussion of the complexities which affect our present method of testing the red-shift, we must also mention the mean absolute magnitude M_0 and the dispersion σ which determine, in accordance with equation (59), the absolute magnitude $\overline{M}$ of a suitable "average" nebula to use in calculating the corrections Δm_1 to the photographic magnitudes. These two quantities, M_0 and σ, must themselves be determined from an analysis of rather limited data on the absolute magnitudes of nearby isolated nebulae, and there is some uncertainty as to how well the distribution can be represented by a Gaussian curve and how accurate the calculated values of the mean and the dispersion may be.

The foregoing long discussion of the difficulties and uncertainties affecting the treatment of nebular counts has seemed necessary in order to emphasize the provisional character of any conclusions as to the nature of the red-shift which can be drawn from such counts

at the present time. The tentative conclusions indicated by an analysis of the five surveys to different magnitudes mentioned above may be stated quite briefly.

In the first place, it is quite certain that the counts cannot be represented by the simple formula (57) which corresponds to stationary objects in Euclidean space without red-shift at all, and that to attain agreement some correction term, Δm_1, which increases with distance, must be subtracted from the limiting magnitudes m_1 as in equation (56). Hence, at least we can say that the red-shift does appear to cut down the luminosity of distant nebulae in the general way that might seem theoretically inevitable. In the second place, nevertheless, it is not so certain that the values of the corrections Δm_1 actually needed to secure agreement with the data are large enough to be regarded as theoretically plausible.[10]

In the case of the non-recessional theory, which predicts smaller effects from the red-shift, approximate agreement with the data could be attained, provided we disregard the possibility of any appreciable light-absorption on the way from the nebulae, and then take for the correction Δm_1 values calculated from equation (58′) with the aid of a K_1 corresponding to the high effective temperature of about 6000°, and at the same time neglect any possible effects of spatial curvature. On the other hand, in the case of the recessional theory, which predicts larger effects from the red-shift, approximate agreement with the data would seem attainable, in accordance with equation (58), only by also making use of possible effects from spatial curvature which correspond to the appearance of the third term in the expression for the correction Δm_1.

In either case the necessity of neglecting any effect of absorption over the long light-path from the nebulae and of getting low values for K_1 by accepting effective temperatures for the nebulae which seem high in view of the present observations on color indices must be regarded with some uneasiness. In addition, in the case of the recessional theory, the necessity for also making use of spatial curvature to cut down the calculated values of the correction Δm_1 must be regarded as in conflict with our usual notions as to the distances to

which observations would have to be carried before appreciable effects from spatial curvature would seem probable.

As to the nature of the spatial curvature, which would be needed in the case of the recessional theory, it is to be noted—as shown by the form of the original equation (5) for the number of nebulae in a given co-ordinate range—that the effects of curvature would be in the right direction to give increased counts at larger distances only for the case of a closed model with R_0 real. This result is of interest in connection with Milne's work, since he has shown[15] that his kinematical model would lead to nebular counts approximating those in a relativistic model with negative curvature, that is, with R_0 imaginary instead of real, as appears observationally indicated. As to the amount of curvature, it is found with the help of the usual relativistic expressions for pressure and density in a homogeneous cosmological model that the necessary value of R_0 would require an averaged-out density in the universe several thousand times as large as that computed from the masses usually assigned to the nebulae. Such a density would necessitate either large amounts of unknown intergalactic material in a form which would not lead to appreciable absorption, or very large masses connected with the nebulae themselves, as is perhaps suggested by the recent work of Sinclair Smith[16] on the mass of the Virgo cluster.

Our conclusions as to the present status of nebular counts can therefore be summarized by stating that it might be possible to explain the results on the basis of either a static homogeneous model with some unknown cause for the red-shift or an expanding homogeneous model with the introduction of effects from spatial curvature which seem unexpectedly large but may not be impossible. This statement must be made very tentatively, however, and may need serious modification in the light of further counts, of new auxiliary data for treating the counts, or of better analysis of the present data.

13. CONCLUDING REMARKS

To bring this long treatment to an end a few further remarks are needed. It is to be noted that both types of explanation for the nebu-

[15] *Relativity, Gravitation and World Structure* (Oxford, 1935), p. 333, eq. (ii‴).

[16] *Pub. A.S.P.*, **47**, 220, 1935.

lar red-shift which we have examined have been treated with the help of models that were assumed to be filled with a uniform distribution of the different kinds of nebulae, having properties sufficiently independent of time to permit the nebulae at all distances being regarded as similar at the time their observed light was emitted.

With regard to the assumed uniformity of nebular distribution, the theoretical possibility for testing the nature of the red-shift by measurements on the surface distribution of nebular luminosity may again be stressed, since the results of such a test should be less dependent on the assumed homogeneity of the model than those derived from nebular counts. As to the possibility of assuming actual deviations from uniform distribution to explain the unexpectedly large nebular counts at great distances, nothing entirely convincing can be advanced against such an assumption. Nevertheless, the analysis already published for the very extensive nebular counts to $m = 19.4$ and $m = 20.0$ shows no evidence for lack of uniformity in different directions. Hence, to make use of non-homogeneity as an explanation in the present connection, we might have to regard ourselves as rather uniquely located near the center of a spherically symmetrical nebular distribution with a density gradually increasing outward, which seems an *ad hoc* hypothesis.

With regard to the assignment of constant properties to the nebulae over long periods of time, it is to be remarked that in the case of the most distant survey, to $m = 21.0$, the "average" nebulae involved must have emitted the light which is now observed at a time of the order of 3×10^8 years in the past. If the recessional explanation of red-shift were adopted, this period would be so nearly comparable with the time of cosmic expansion—possibly of the order of 10^9 to 10^{10} years—that the assignment of constant properties might not appear justified, and in that case we might try to explain excess counts by assuming greater luminosities for the nebulae at earlier times. On the other hand, with the non-recessional theory, there would be nothing to indicate a short-time scale for nebular evolution, which indeed might seem to be an advantage for that theory.

In conclusion it may be repeated that the main function of the present article has been to provide a formulation for the two theo-

retical methods of investigating the nature of the red-shift which would be applicable under actual circumstances, the length of the article being due to the real complexity of these circumstances. In addition it seemed of interest to give some tentative statement as to the present status of the data for the tests, even though their complete presentation and analysis must be reserved for a later time. It also seemed desirable to express an open-minded position as to the true cause of the nebular red-shift, and to point out the indications that spatial curvature may have to play a part in the explanation of existing nebular data.

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A QUANTITATIVE STUDY OF CERTAIN PHASES OF F-TYPE SPECTRA

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ABSTRACT

Quantitative data are given concerning the behavior of a number of the more important lines in F spectra. These lines include members of many of the most important pairs used as luminosity criteria for the determination of spectroscopic parallaxes in the classical researches of the Mount Wilson Observatory. A discussion is given of the behavior of the various lines in each F subclass.

The sharp- and diffuse-line F stars.—A parameter (line width/ line depth) is given as a measure of the diffuseness of the metallic lines exhibiting various degrees of rotational broadening. This parameter varies over a range of more than thirty times among stars of Draper type Fo.

A comparison of Harvard and Mount Wilson spectral types shows that practically all of the bright diffuse-line Fo stars have been moved over into class A by Mount Wilson. The mean Mount Wilson spectral type of the twenty-one most diffuse H.D. Fo spectra is A 5.6.

The intensities of *Ca* II K and of *Hδ* in sharp- and diffuse-line Fo stars confirm the consistency of the Henry Draper classification; at type Fo there is no systematic difference in the intensities on passing from narrow- to broad-line stars. There is a tendency for the strong *Fe* I lines to be systematically fainter in the diffuse-line stars; it is possible, however, that this is due to systematic errors in measuring the equivalent widths.

The sharp drop in the number of rapidly rotating stars between Fo and F5 is confirmed, and it is shown that the diffuse-line stars are distributed over a considerably smaller range of spectroscopic absolute magnitude than are those having sharp lines. For sharp-line Fo stars the range is approximately 5 mag.; for twenty-one of the most diffuse-line Fo stars it is 0.9 mag., the latter clustering about absolute magnitude +2.0. The effect is not so pronounced when trigonometric absolute magnitudes are used, and it is possible that the difference is a result of a systematic error in the determination of spectroscopic parallaxes for rapidly rotating stars.

The behavior of Ca II *K and of Hδ with changes in spectral type.*—A brief quantitative discussion of some of the most prominent F-type classification criteria is given. *Ca* II K increases in intensity about 50 per cent from Fo to Go and *Hδ* decreases 75 per cent in the same range. The ratio K/*Hδ* increases from 0.7 at Fo to 3.8 at Go. Visual estimates of this ratio are usually given as considerably greater; this difference between measured and visual estimates is ascribed to the effect on the estimates of the Stark broadening of *Hδ*.

The behavior of the luminosity criteria.—Equivalent widths in approximately one hundred F stars are given for the individual lines of five line ratios which have been used in the determination of spectroscopic luminosities at Mount Wilson. A quantitative summary is given of the behavior of these line ratios with luminosity in the F subdivisions. At type F5 the most sensitive criteria among those discussed are: *Fe* II—*Fe* I 4233/*Fe* I 4235 and *Sc* II 4246/*Fe* I 4250; at F8: *Fe* I 4071/*Sr* II 4077 and *Ti* II—*Zr* II 4161/*Mg* I 4167; at Go: *Fe* I 4071/*Sr* II 4077 and *Sr* II 4215/*Fe* I 4250.

The variation of Ca I *and Fe* I *with luminosity at F8 and Go.*—The lines *Ca* I 4226 and *Fe* I 4045 vary with luminosity in a peculiar manner at types F8 and Go. Both lines are very strong at $M=+4.5$. They then decrease in intensity to $M=+3.3$. In the giants, however, both lines are strengthened with increasing luminosity. In the case of *Ca* I 4226 the intensity reaches a maximum and is identical in dwarfs ($M=+4.5$) and in the most luminous giants ($M=-3.5$). For intermediate luminosities the line has only one-third of its maximum intensity. Its range in equivalent width is from 1.5 to 0.5 A.

A possible explanation of the phenomenon is given. In the dwarfs the expected negative absolute-magnitude effect is observed, owing to the depletion of *Ca* and *Fe* atoms in the neutral state caused by the lowered pressure in the stellar atmospheres. In the giants, however, the tremendously greater extent of the atmospheres and the strengthening of all lines counterbalances and masks the effect observed in the dwarfs.

The intermediate giants of type Go.—It is unlikely that the three Go stars η Pegasi, o Ursae Majoris, and β Leporis, which have been assigned spectroscopic absolute magnitudes around +0.5 at Mount Wilson, are less luminous than the ordinary Go giants. Their trigonometric parallaxes, as well as their spectroscopic luminosity criteria, fail to support the low luminosities found at Mount Wilson.

I. INTRODUCTION

1. As the determination of spectroscopic parallaxes of stars later than class A has reached a very satisfactory state of accuracy at the present time, it is of some importance to investigate quantitatively the criteria used in the determination of luminosities spectroscopically. The present investigation is concerned with a study of some of the more important standards of absolute magnitude in the stars of types Fo–Go. The classical work which has been in progress at the Mount Wilson Observatory for the last twenty years has been taken as the basis for discussion.

The problem of spectral classification is so intimately bound up with that of the determination of spectroscopic parallaxes that a discussion of either subject must of necessity include the other also. In addition, a quantitative discussion will be given of certain features of the classification of the F stars.

2. The observational material consists of 440 spectrograms of 201 stars brighter than visual magnitude 5.5 and situated north of declination $-20°$. The plates are of one-prism dispersion and were obtained at the Yerkes and the Perkins observatories; the scale of the spectrograms is approximately 25 A/mm at $H\gamma$. The Yerkes and Perkins plates have almost identical dispersions and were obtained on similar emulsions, the spectrograms measured having been obtained on Eastman Hyper-Press and Eastman 40 plates. The total absorptions of a selected group of lines were measured in 56 of the stars, and estimated intensities for these and for 54 additional stars were calibrated by means of the measures. The spectra of the additional 91 stars had been obtained a number of years ago on a variety of emulsions, and as they were not usually calibrated and are not strictly comparable, they were used only in certain qualitative aspects of the work.

The spectrograms were calibrated with tube sensitometers. Although the calibration is not monochromatic, the plate contrast does not change appreciably over the range in wave-length included, and a comparison of plates taken on the Eastman 40 emulsion with those taken on the orthochromatic Hyper-Press emulsion showed that no sensible errors are introduced by the type of calibration employed.

Two microphotometers were used for the measurement of the plates: the first, that belonging to the Mendenhall Physical Laboratory of Ohio State University; the second, that of the Ryerson Physical Laboratory of the University of Chicago. Both are of the Moll type. In the case of the line $H\delta$, actual profiles were obtained for the determination of the total absorptions; a simplified method of reduction was used for the other lines. The total absorptions were computed from the measured bases and line depths on the assumption that the profiles are triangular in shape. It has been shown that no appreciable error is introduced by this simplification. The probable error of a single calibrated estimate of total absorption is approximately 10 per cent, except in the case of weak lines.

II. THE SHARP- AND DIFFUSE-LINE F STARS

3. Systems of spectral classification which are not based on the Henry Draper system are little utilized at the present time and will not be discussed. The Pickering-Fleming classification of 1890[1] laid the foundation for the system which was later refined and is now universally used. The detailed description of the classification of the F stars published by Miss Cannon[2] in 1900 has remained sensibly unchanged up to the present time, as far as classification from objective-prism spectra is concerned. During the last twenty years there have been, however, a number of reclassifications in connection with work on the determination of spectroscopic parallaxes, called forth mainly because of the use of increasingly higher dispersion. The principal reclassifications which have been made are those of Adams and his collaborators,[3] of Abetti,[4] and of Young and Harper.[5] In all of these cases an attempt was made to adhere as closely as possible to the Henry Draper system.

[1] *H.A.*, **27**, 1, 1890.
[2] *Ibid.*, **28**, 146, 1900.
[3] *Ap.J.*, **53**, 13, 1921.
[4] *Pub. Obs. Arcetri*, **41**, 9, 1924; **42**, 9, 1925.
[5] *Dom. Ap. Obs.*, **3**, 1, 1927.

Two systems of reclassification were used in the earlier Mount Wilson work. The so-called "measured" class was determined from ratios of intensities of $H\beta$ and $H\gamma$ to neighboring lines. As these ratios also have a pronounced absolute-magnitude effect in certain cases, their determination was later abandoned. The second system gives the so-called "estimated" classes. These were determined by comparison with typical spectra of the various subdivisions. The K line of *Ca* II, whose intensity ratio with respect to $H\delta$ is the principal Harvard criterion in the range A0–F5, was not available on Mount Wilson plates. For the A and F0 stars the Mount Wilson classification is based on the intensities of the metallic arc lines. If the strong *Fe* I lines λ 4325 and λ 4383 were absent or very faint, the spectrum was considered to be of type A0. As the arc lines increased in intensity, the spectra were considered to belong to successively later subdivisions. An attempt at refinement was made by utilizing all the decimal subdivisions available. As the Mount Wilson classification of the A and early F stars depends on the visibility and absolute intensity of comparatively faint lines, a systematic difference in classification between stars having sharp and diffuse lines as compared with Harvard types might be expected. Such a difference has been found for the A stars by Lindblad[6] and, for the B, A, and F stars by Miss Fairfield.[7] Miss Fairfield has also shown that the Mount Wilson diffuse-line classification does not parallel the Henry Draper system. The reclassification by Abetti rests on standards determined at Mount Wilson and is, in general, similar to the Mount Wilson results. The Victoria criteria were the same as those given most weight at Harvard. As a consequence, it would be expected that the classification by Young and Harper would differ little from the Draper system. From the fact that considerably more stars have been included in the Mount Wilson investigations than in the other series of spectroscopic parallaxes, the discussion of the classification of the F stars will be concerned principally with the Harvard and Mount Wilson data.

4. The most striking dissimilarities in the appearance of the spectra of the early F stars are those due to differences in the widths of the lines of medium and low intensity. These differences have been

[6] *Ap.J.*, **59**, 305, 1924. [7] *H.C.*, **264**, 1924.

shown by Struve, Shajn, and Elvey[8] to be an effect of varying speeds of axial rotation of the stars. The line widths of all stars for which plates were available were estimated on an arbitrary scale of one to four, group IV including the stars having the broadest lines. The estimated widths were then calibrated from microphotometric measures of *Fe* I 4045 in 50 stars. The ratio of the line width in angstroms to the line depth in magnitudes (W/m) was taken as the most convenient parameter. Table I gives the mean values of (W/m) for the four

TABLE I

Estimated Width	Mean (W/m)	No. of Stars
I	3.0	23
II	5.1	14
III	8.9	8
IV	17.8	5

TABLE II

Draper Class	Line Class			
	I	II	III	IV
Fo	17	18	21	23
F5	23	22	3	1
F8	15	9	0	0

classes of estimated line widths. The range in extreme cases is more than thirty times.

Table II shows the distribution of the diffuse-line stars in Harvard classes Fo, F5, and F8. As there are only about one-fourth as many F2 stars as those of type Fo or F5 among the bright stars classified at Harvard, class F2 has not been included. It is apparent that the relative number of broad-line stars decreases rapidly with advancing spectral type. No diffuse-line stars have been observed in types F8 and Go. The rapid falling-off in distribution was also shown by Miss Westgate[9] from visual measures of line widths.

[8] *M.N.*, **89**, 222, 1929; *Ap. J.*, **70**, 141, 1929; *Ap.J.*, **72**, 1, 1930.

[9] *Ibid.*, **79**, 357, 1934.

A comparison of the Harvard and Mount Wilson classifications of stars belonging to the four line classes is given in Table III. There is a pronounced systematic difference at type Fo in the Harvard and Mount Wilson classifications for sharp- and diffuse-line stars. Practically all the Harvard Fo stars of line width IV have been moved over into class A by Mount Wilson. There is a tendency in the same direction at F5, but the number of broad-line stars is very small.

5. In view of the marked systematic difference between the Harvard and Mount Wilson classifications at Fo, it is of interest to in-

TABLE III

Spectral Type (Harvard)	Line Class	Mean Spectral Type (Mount Wilson)	No. of Stars
Fo	I	F 1.2	16
	II	F 1.5	18
	III	A 8.2	20
	IV	A 5.6	21
F5	I	F 4.4	22
	II	F 4.1	22
	III	F 2.3	3
	IV	A 7.0	1
F8	I	F 7.0	13
	II	F 7.8	11
Go	I	G 0.7	19
	II	G 0.2	9

vestigate the behavior of *Ca* II K and *Hδ*, the ratio of whose intensities is the most important Harvard standard of spectral type among the A and early F stars. Table IV gives the intensities of K as a function of Harvard spectral type and line class. It is apparent from the tabulated intensities that no appreciable systematic difference in the intensity of *Ca* II K exists among stars of different line widths of the same Harvard class. This is a confirmation of the consistency of the Henry Draper classification for sharp- and diffuse-line stars.

The intensities of *Hδ* are given in Table V. Measures of *Hδ* are more uncertain than those of the K line because of the changing

shape of the profile as a result of Stark effect. There is also a real dependence of the intensity of $H\delta$ on luminosity. Table V shows, however, that at type Fo (Harvard) there is no appreciable correlation of total absorption with line class. The mean of the diffuse-line classes III and IV is almost identical with that of the sharp classes I

TABLE IV

INTENSITIES OF *Ca* II K

Harvard Spectral Type	Line Class	Total Absorption (Angstroms)	No. of Stars
Fo	I	5.1	9
	II	4.7	8
	III	5.2	7
	IV	5.4	6
F5	I	5.4	12
	II	5.2	8
	III	5.1	3
F8	I	6.9	10
	II	6.0	5
Go	I	7.0	13
	II	8.1	2

TABLE V

INTENSITIES OF $H\delta$

Harvard Spectral Type	Line Class	Total Absorption (Angstroms)	No. of Stars
Fo	I	7.0	6
	II	7.0	4
	III	5.7	6
	IV	8.4	6
F5	I	4.1	6
	II	4.8	5
	III	6.4	3
F8	I	2.9	5
	II	3.3	1
Go	I	1.9	8

and II. At type F5 there seems to be a tendency for the more diffuse-line stars to have higher intensities for *Hδ*. As the number of broad-line stars is very small at F5, this tendency should not be considered as definite. It can be concluded, therefore, that the behavior of *Hδ* is also in agreement with the Harvard classification. As the *Fe* arc lines form one of the principal criteria of spectral type in the Mount Wilson classification of the A and Fo stars, it is of importance to investigate their behavior with respect to line width. Table VI gives the intensities of the unblended lines of the strong *Fe* I multiplet $a^3F - y^3F^\circ$ at H.D. class Fo.

TABLE VI

INTENSITIES OF *Fe* I LINES

(H.D. Class Fo)

Line Class	Intensity (Angstroms)				No. of Stars
	λ 4045	λ 4063	λ 4071	λ 4132	
I	0.58	0.47	0.30	0.31	10
II	.59	.46	.29	.25	9
III	.42	.31	.32	.24	6
IV	0.36	0.30	0.24	0.17	5

The *Fe* I lines seem to be systematically weaker in the diffuse-line Fo stars than in those having sharp lines. If this effect is real, it would justify the placing of diffuse-line Draper Fo stars in class A by Mount Wilson. There is a possibility, however, that errors in the intensities due to systematically drawing in the continuous background too low on the microphotometer tracings increase with increasing diffuseness of the successive width classes. It should be noted that the determinations of the intensities of K and of *Hδ* are affected little, if at all, by the varying degree of diffuseness so apparent in the weaker lines. It may therefore be stated that the results, in general, confirm the consistency of the Draper classification.

It is interesting to compare the distribution of sharp- and diffuse-line stars with absolute magnitude. Table VII shows the total range in spectroscopic absolute magnitude for the four groups of line widths in Harvard Fo stars, the only F class in which there is an abundance of diffuse-line stars.

Table VII shows clearly a progressive decrease in the range of spectroscopic luminosity with increasing diffuseness of lines in the Harvard Fo stars. Trigonometric absolute magnitudes are not reliable for the early F giants, but indicate that the same phenomenon occurs, though in lesser degree, when the stars included in Table VII are plotted against luminosities determined trigonometrically. If the effect is considered real, then the small range in luminosity of the diffuse-line stars, and their strong preference for an absolute magnitude of about +2, remains unexplained. It seems more probable, however, that the extreme diffuseness of the lines in these stars makes estimation of the luminosity criteria very difficult, and that the effect noted might be considered as a systematic error in the assignment of spectroscopic absolute magnitudes.

TABLE VII

Line Class	High M	Low M	Range ΔM	No. of Stars
I	−2.0	+3.2	5.2	14
II	+0.2	3.3	3.5	18
III	+1.1	2.9	1.8	20
IV	+1.4	+2.3	0.9	21

Since the diffuse-line stars cannot be compared directly with those having sharp lines without danger of introducing systematic errors in the measures, it has been considered advisable to include only stars of line-width class I in the following discussion. This class comprises the stars in which no broadening of the spectral lines can be detected on one-prism plates.

III. THE BEHAVIOR OF *Ca* II K AND OF *Hδ* WITH CHANGES IN SPECTRAL TYPE

6. From the measured intensities of *Ca* II K and of *Hδ* a quantitative investigation of the behavior of these standards according to spectral type can be made. Table VIII gives the mean intensities of K and of *Hδ* for Harvard classes Fo, F5, F8, and Go. The number of stars in which each line was measured can be found by referring to Tables III and IV. K is approximately 50 per cent stronger, and *Hδ* is about one-fourth as strong at Go as at Fo. The ratio K/*Hδ* in-

creases from 0.7 at Fo to 3.8 at Go. The visual ratio at Fo is given by Miss Cannon as about 3.0.[2] The discrepancy is probably due to the different shape of the two lines. The broader wings of $H\delta$ do not influence estimated intensities of the line as much as their actual contribution to the total absorption would demand. Most of the change in the ratio $K/H\delta$ is due to the great decrease in intensity of $H\delta$ in the later subclasses.

TABLE VIII

Draper Class	Total Absorption (Angstroms)		Ratio $K/H\delta$
	K	$H\delta$	
Fo	5.1	7.0	0.7
F5	5.3	4.8	1.1
F8	6.6	3.0	2.2
Go	7.2	1.9	3.8

TABLE IX

Ratio	Numerator	Denominator	Observatory
4071/4077	*Fe* I	*Sr* II	Mount Wilson, Norman Lockyer, Victoria
4161/4167	*Zr* II, *Ti* II	*Mg* I	Mount Wilson, Victoria
4215/4250	*Sr* II	*Fe* I	Mount Wilson, Victoria, Norman Lockyer
4233/4236	*Fe* II, *Fe* I	*Fe* I	Mount Wilson
4246/4250	*Sc* II, *Fe* I	*Fe* I	Victoria

IV. THE BEHAVIOR OF THE LUMINOSITY CRITERIA

7. The individual components of five of the most important line ratios which have been used to determine spectroscopic parallaxes among the F-type stars were selected for quantitative investigation. The line ratios investigated are listed in Table IX.

The intensities of ten individual lines are listed in Table X as a function of spectral type.

The *Fe* I lines increase gradually and fairly uniformly in intensity on passing to stars of lower temperatures. The mean intensity of λ 4250 at Go is possibly an exception. The ultimate *Sr* II doublet

λ 4077 and λ 4215 also increases gradually in intensity in the later subdivisions. There is some evidence that the rate of increase in intensity with decreasing temperature is greater in the case of λ 4215. The blended line at λ 4161, which is due almost equally to *Zr* II and to *Ti* II, shows little change with temperature. The line at λ 4226, which is due principally to *Ca* I, shows a marked increase in intensity in successively later spectral subdivisions. There is no appreciable change in *Sc* II 4246 with changing temperature. λ 4233 is due principally to *Fe* II in the early F stars and about equally to *Fe* I and *Fe* II at F8 and Go. *Fe* I 4235 increases in intensity uniformly through successively later subdivisions. The lines least affected by changes in temperature are *Sr* II 4077, *Mg* I 4167, and *Sc* II 4246.

TABLE X

Draper Class	Intensity in Angstroms									
	4071	4077	4161	4167	4215	4226	4233	4236	4246	4250
Fo	0.31	0.51	0.28	0.32	0.46	0.54	0.38	0.28	0.30	0.32
F5	.44	.51	.35	.32	.50	0.66	.44	.34	.37	.35
F8	.53	.60	.39	.37	.65	1.02	.58	.46	.44	.48
Go	0.63	0.63	0.35	0.44	0.79	1.21	0.55	0.51	0.32	0.39

8. The behavior of the lines with respect to changes in luminosity is shown in Figures 2–6, in which the total absorptions are plotted against luminosities as determined from trigonometric parallaxes taken from the new *Yale Catalogue*. In the case of definite supergiants, such as ε Aurigae, φ Cassiopeiae, γ Cygni, and δ Canis Majoris, an absolute magnitude of −4 has been assumed. For other giants whose trigonometric parallaxes are insensible a parallax of 0.008 was assumed.

a) *Fe* I 4071:*Sr* II 4077 (Fig. 1)

There is no very marked absolute magnitude effect for *Fe* I 4071 at any place among the F stars. It is possible that the line is slightly strengthened on passing from dwarfs to giants at type Go. The remarkably high intensity of *Sr* II 4077 in certain Fo dwarfs such as

γ Equulei is abnormal, as there is a definite increase in intensity in most giants as compared with the dwarfs. There is a uniform in-

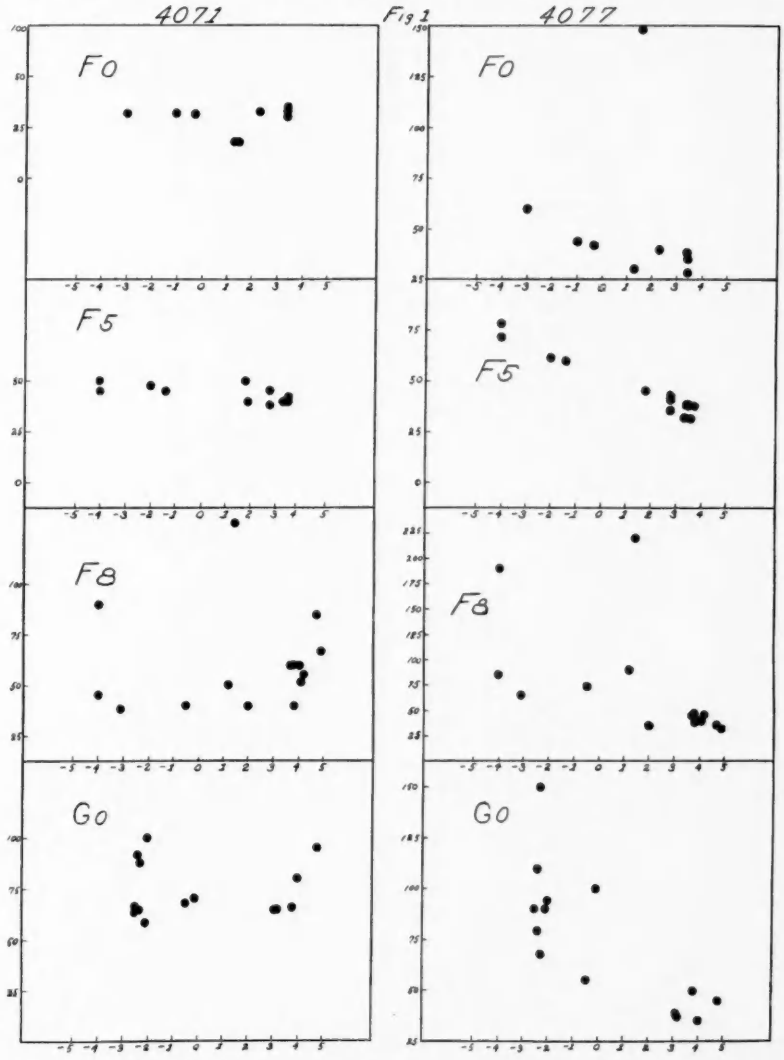

FIG. 1.—Intensities of *Fe* I 4071 and *Sr* II 4077. Abscissas are absolute magnitudes; ordinates are equivalent widths in units of 0.01 A.

crease in the intensity of λ 4077 on passing toward the more luminous stars in classes F5, F8, and G0. The two discordant points at

class F8 refer to ρ Cassiopeiae and 11 Puppis. All details of their spectra point toward very high luminosity, but each star has a rather definitely determined trigonometric parallax.

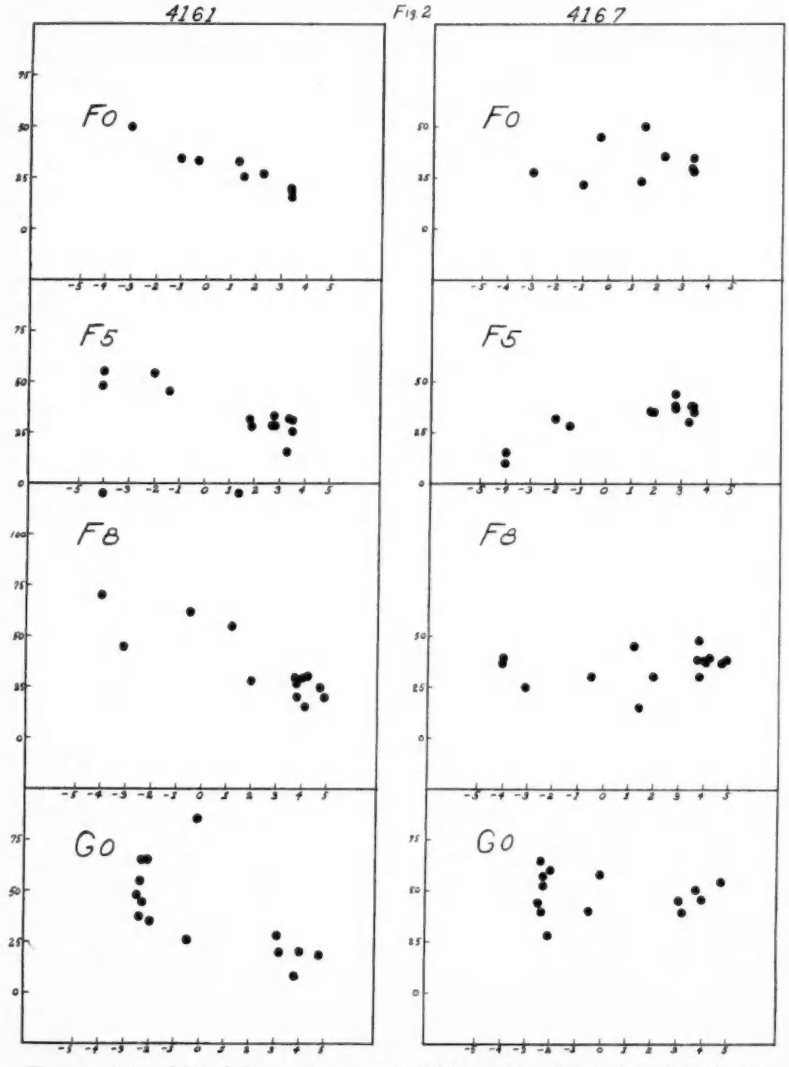

FIG. 2.—Intensities of *Ti* II—*Zr* II 4161 and *Mg* I 4167 as a function of absolute magnitude.

As *Fe* I 4071 is little affected by differences in luminosity, changes in the ratio 4071/4077 are due almost entirely to changes in the de-

nominator. Table XI gives smoothed values of intensity of λ 4077 and of the ratio 4071/4077 for stars of absolute magnitude −2 and +3.5.

TABLE XI

Draper Class	Intensity				Ratio 4071/4077	
	λ 4077		λ 4071			
	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$
Fo	0.57	0.33	0.32	0.32	0.56	0.97
F5	.61	.35	.43	.41	.70	1.17
F8	.82	.38	.50	.50	.61	1.32
Go	0.82	0.37	0.66	0.58	0.81	1.57

Sr II 4077 shows a considerable increase in intensity for stars having $M=-2$, in passing toward later spectral classes. In the dwarfs, however, there is no appreciable change in intensity throughout the F-type subdivisions.

b) *Ti* II—*Zr* II 4161 : *Mg* I 4167 (Fig. 2)

There is a definite luminosity effect for λ 4161 in all subdivisions of the F stars, the line tending to increase in strength with increasing luminosity. The intensity of *Mg* I 4167 is relatively insensitive to

TABLE XII

Draper Class	Intensity				Ratio 4161/4167	
	Ti II—*Zr* II 4161		*Mg* I 4167			
	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$
Fo	0.50	0.19	0.30	0.30	1.70	0.63
F5	.50	.26	.28	.33	1.80	.79
F8	.67	.27	.33	.39	2.00	.69
Go	0.45	0.18	0.44	0.40	1.00	0.45

differences in absolute magnitude. There is a possible tendency at Fo and at F8 for the line to be weaker in giants than in dwarfs. At F8, ρ Cassiopeiae again gives contradictory results. Table XII summarizes the behavior of the two lines individually and of their ratio.

The blended enhanced line λ 4161 passes through a maximum intensity at F8 both in the giants and in the dwarfs. *Mg* I 4167 shows

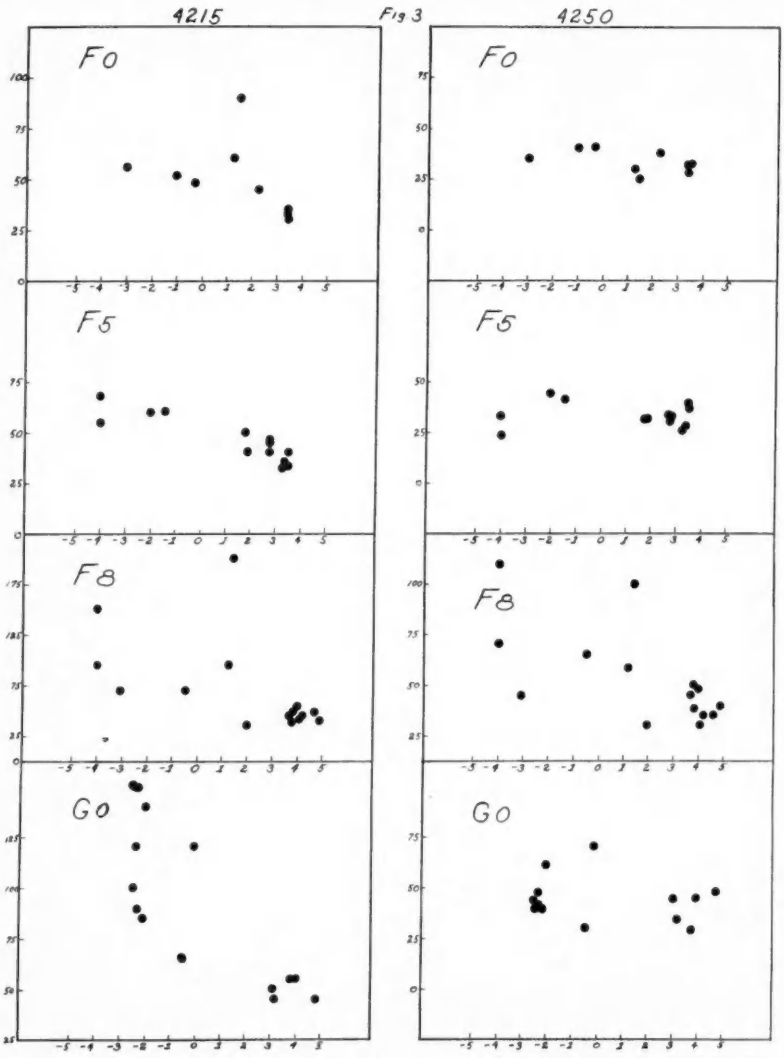

FIG. 3.—Intensities of *Sr* II 4215 and *Fe* I 4250 as a function of absolute magnitude.

little regular variation with temperature in the giants, although there seems to be a slight increase in intensity in passing to the later subdivisions.

c) *Sr* II 4215:*Fe* I 4250 (Fig. 3)

With the exception of a few "strontium dwarfs" there seems to be a good correlation of λ 4215 with absolute magnitude at Draper class Fo for stars absolutely fainter than $M = +1$. The variation of *Sr* II 4215 with absolute magnitude is also well defined at F5, F8, and Go. There is, however, a considerable scatter among the F8 and Go giants.

Table XIII gives the smoothed mean intensities and the intensity ratios of λ 4215 and λ 4250.

TABLE XIII

Draper Class	Intensity				Ratio 4215/4250	
	Sr II 4215		*Fe* I 4250			
	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$
Fo	0.55	0.30	0.32	0.32	1.72	0.94
F5	0.60	.35	.32	.32	1.87	1.09
F8	0.90	.50	.65	.40	1.38	1.25
Go	1.00	0.55	0.42	0.35	2.38	1.31

TABLE XIV

Draper Class	Intensity				Ratio 4233/4235	
	Fe II—*Fe* I 4233		*Fe* I 4235			
	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$
Fo	0.58	0.27	0.32	0.25	1.8	1.1
F5	.60	.28	.33	.33	1.8	0.9
F8	.75	.45	.65	.43	1.2	1.0
Go	0.60	0.40	0.53	0.40	1.1	1.0

Sr II increases uniformly in intensity with the later types, for both the giants and the dwarfs. *Fe* I 4250 increases slightly in the same direction in the dwarfs, but there are no definite conclusions to be drawn from its behavior in the giants.

d) *Fe* II—*Fe* I 4233:*Fe* I 4235 (Fig. 4)

The intensities and the ratios of this pair are given in Table XIV.

λ 4233 passes through a maximum of intensity both in the giants and in the dwarfs at F8. As both lines of this pair tend to increase in

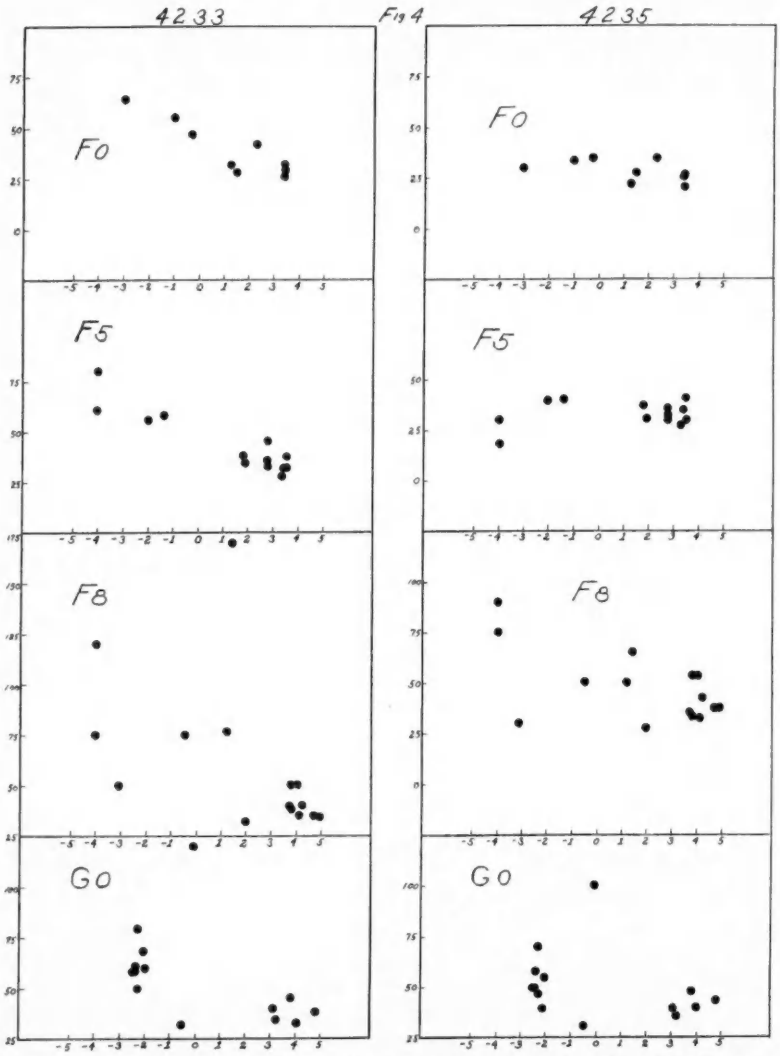

FIG. 4.—Intensities of *Fe* II—*Fe* I 4233 and *Fe* I 4235 as a function of absolute magnitude.

intensity with increasing luminosity, their ratio is not a very sensitive criterion for the determination of absolute magnitude except at type F5.

e) *Sc* II 4246:*Fe* I 4250 (Fig. 5)

The intensities and the ratios of the pair are given in Table XV.

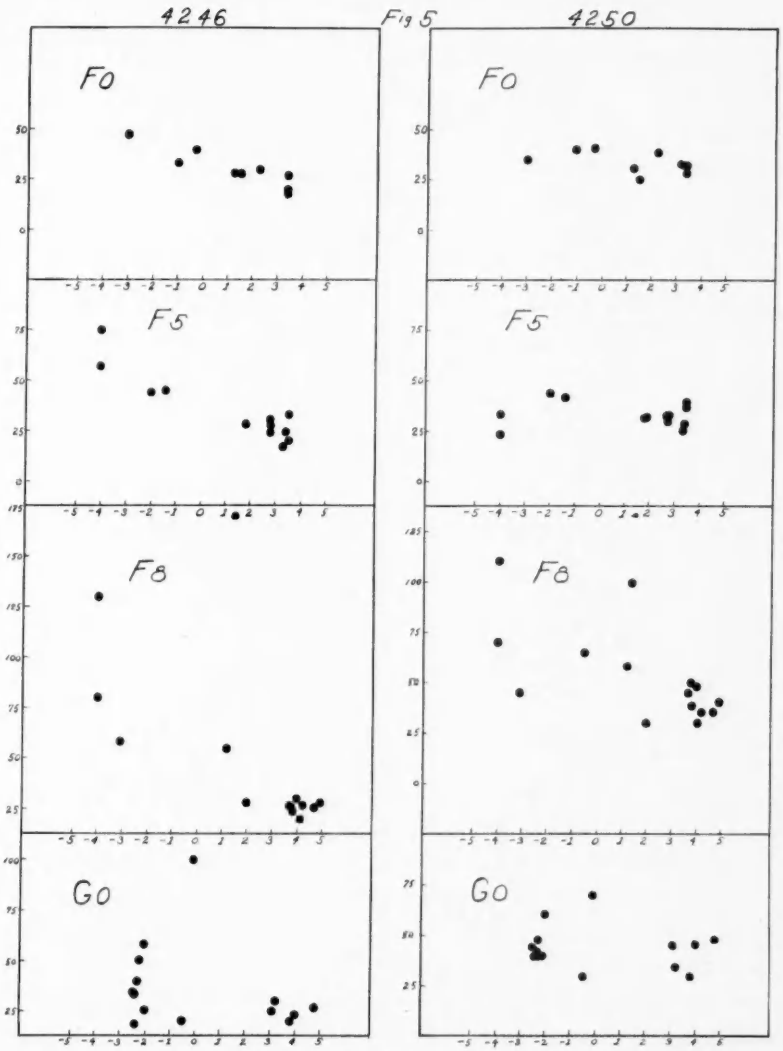

FIG. 5.—Intensities of *Sc* II 4246 and *Fe* I 4250 as a function of absolute magnitude.

Sc II 4246 appears to have a maximum at about F8, pronounced in giants but not appreciable in the dwarfs. The behavior of the iron line at λ 4250 has already been discussed. The ratio of these two

lines shows the greatest range at F5, and at Go it ceases to be appreciably influenced by luminosity.

We may summarize the spectral subdivisions in which each line ratio shows the greatest range with luminosity. This is shown in Table XVI.

TABLE XV

Draper Class	Intensity				Ratio 4246/4250	
	Sc II 4246		*Fe* I 4250			
	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$	$M=-2$	$M=+3.5$
Fo	0.41	0.22	0.32	0.32	1.3	0.69
F5	.50	.22	.32	.32	1.6	.69
F8	.75	.32	.65	.40	1.1	.80
Go	0.38	0.28	0.42	0.35	0.90	0.80

TABLE XVI

Pair	Spectral Type Showing Greatest Effect (Draper)
4071/4077	F8, Go
4161/4167	F8
4215/4250	Go
4233/4235	F5
4246/4250	F5

V. THE BEHAVIOR OF *Ca* I 4226 AND *Fe* I 4045 AT F8 AND Go

9. The strongest line of *Fe* I in the observed spectral range at λ 4045 shows no very striking behavior with changes in luminosity in types earlier than F8, but at that type an actual change of sign in absolute-magnitude effect takes place on passing from the dwarfs to the giants. The same effect is clearly marked for the ultimate line of *Ca* I at λ 4226. The intensities of both lines are plotted against Mount Wilson spectroscopic absolute magnitudes in Figure 6. The intensity of *Ca* I 4226, which is 1.5 A at $M=+4.5$, decreases to 0.5 A at $M=+3.3$. No stars of intermediate luminosity are observed, and at the next observed point, $M=-2.0$, the intensity is 0.7 A. On passing to the giants of highest luminosity the line increases in intensity to 1.5 A at $M=-3.5$. The intensity is thus identical at $M=+4.5$ and $M=-3.5$, a range of 7 mag. in luminosity.

The same phenomenon is shown for *Fe* I 4045, except that the line never becomes as strong in the dwarfs as it is in the giants of the highest luminosity.

VI. THE INTERMEDIATE GIANTS OF TYPE Go

The behavior of *Ca* I 4226 and *Fe* I 4045 at Go is shown in Figure 7. The same effect is present, although it is partially masked by three of the "intermediate giants," which are shown as open circles. The existence of this intermediate group of stars was shown statistically by Strömberg,[10] and special reduction curves were used at

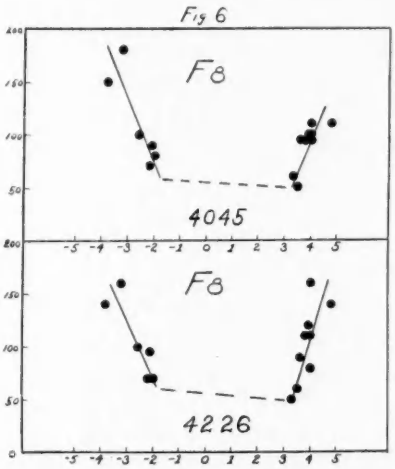

FIG. 6.—Intensities of *Fe* I 4045 and *Ca* I 4226 as a function of luminosity at type F8.

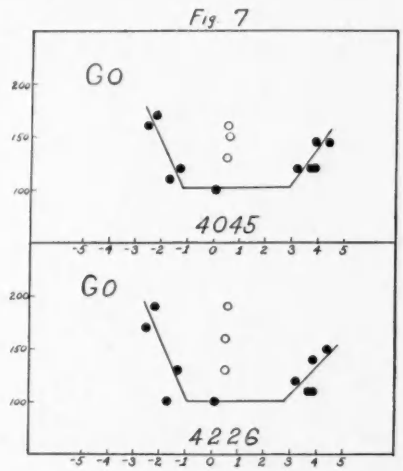

FIG. 7.—Intensities of *Fe* I 4045 and *Ca* I 4226 as a function of luminosity at type Go.

Mount Wilson for deriving their absolute magnitudes spectroscopically. It is of considerable importance to ascertain whether these three stars have luminosities definitely less than those of the normal giants. Table XVII gives a comparison of the three "intermediate giants" (group I) and the four giants proper (group II) at Go.

A comparison of the spectroscopic and trigonometric parallaxes of the three stars in group I leaves little doubt that the spectroscopic parallaxes are too large. In group I the mean of the spectroscopic parallaxes is 0".030 and the mean of the trigonometric parallaxes is 0".004; in group II the corresponding values are 0".009 and

[10] *Ibid.*, **78**, 178, 1933.

0".006. From the trigonometric data it seems fairly certain that the stars of group I are in general as luminous as those of group II. This is in agreement with the behavior of *Fe* I 4045 and *Ca* I 4226. The three stars of group I show spectroscopic evidence of considerably higher luminosity than Capella ($M = -0.5$ [trig.]). If these objects are moved over so that they agree with the normal giants, then the peculiar absolute-magnitude effect found at type F8 is well shown at Go also.

TABLE XVII

LUMINOSITIES OF Go GIANTS AND INTERMEDIATES

Star	m	M (M.W.)	π (Spec.)	π (Trig.)
	Group I			
η Peg	3.1	+0.6	0".032	−0".002 ± 7
o UMa	3.5	+ .5	.025	+ .005 ± 6
β Lep	3.0	+0.5	0.032	+0.009 ± 6
	Group II			
ϵ Leo	3.1	−1.3	0".011	+0".003 ± 6
β Dra	3.0	−1.7	.013	+ .002 ± 8
α Aqr	3.2	−2.2	.008	+ .004 ± 5
β Cam	4.2	−2.5	0.005	+0.014 ± 11

A possible explanation of the behavior of *Fe* I 4045 and *Ca* I 4226 may be suggested. The weakening of the lines along the dwarf sequence when passing toward higher luminosities may be the expected ionization effect. Neutral lines will be weakened with relation to singly ionized lines because of the decreased pressure in the stellar atmospheres. When the giant stars are reached, however, the atmospheres become so tremendously extensive that all lines are strengthened because of the additional amount of material effective in absorbing the spectral lines. This positive effect is so pronounced that it overcomes the negative effect which determines the behavior of the neutral lines in the dwarf sequence.

Table XVIII gives in successive columns: (1) the serial number; (2) the star's name; (3) the trigonometric parallax; (4) the probable

TABLE XVIII

No.	Star	Trigonometric			Spectroscopic Absolute Magnitude				Spectral Type		
		π	P.E.	Mag.	M.W.	Vic.	N.L.O.	Arc.	M.W.	Vic.	Arc.
					Harvard Fo						
Width I:											
1	σ Boo	0".062	.006	3.4	3.2	.....	2.9	....	F2	.....	.....
2	γ Vir(S)	89	7	3.4	2.8	3.2	2.9	....	Fo	F3	.....
3	γ Vir(N)	89	7	3.4	2.9	2.9	.....	....	Fo	F1	.....
4	9 Aur	.....	.....	.....	2.5	.....	2.9	....	F3	.....	.....
5	η Lep	50	7	2.3	2.8	.....	2.4	....	F2	.....	.....
6	γ Equ	22	6	1.5	−0.2	.....	.....	....	F1	F1	Fos
7	β Cor	33	5	1.3	1.1	−0.3	2.1	1.7	A7s	Fo	A6s
8	20 CVn	10	6	−0.3	2.0	0.8	−1.4	....	F5	F3	.....
9	ν Her	2	5	−1.0	2.0	1.4	.....	....	F1	F3	.....
10	α Lep	− 3	6	−3.0	−0.9	−1.6	−0.8	....	F3	F4	.....
Width II:											
11	η Crv	56	8	3.2	2.9	.....	3.0	....	F2	.....	.....
12	ρ Gem	57	6	3.0	2.4	2.4	2.0	2.4	A8s	A8	A9n
13	7 And	48	7	3.0	2.7	2.6	2.8	....	F2	F1	.....
14	18 Boo	.....	.....	.....	3.0	.....	2.5	2.3	F1	....	F8n
15	ω Oph	33	8	2.2	1.2	.....	1.5	....	A6s	.....	.....
16	30 LMi	.....	.....	.....	2.1	2.0	2.2	....	F3	F1	.....
17	μ Cet	31	8	1.8	2.2	.....	2.3	....	F4	.....	.....
18	γ Cap	28	7	1.0	1.8	.....	.....	....	F2	.....	.....
19	22 And	−0.006	.010	−0.4	0.2	.....	1.9	....	cF2	.....	.....
Width III:											
20	16 Lib	0".040	.005	2.6	1.8	.....	2.1	2.1	A9n	....	F1s
21	38 Gem	39	6	2.6	2.1	.....	2.8	2.0	A7n	....	A6n
22	τ Cyg	50	5	2.3	2.7	2.0	2.2	2.4	Fo	F1n	Fon
23	δ Aql	54	6	2.1	1.9	.....	2.4	....	A5n	.....	.....
24	o Gem	16	6	1.0	2.2	0.7	1.3	2.1	F1	F1	F1sn
25	ζ Leo	10	6	−1.4	1.3	0.6	0.3	2.2	Fo	F1	F2sn
Width IV:											
26	δ Gem	60	5	2.4	2.2	.....	1.8	....	A8n	.....	.....
27	78 UMa	30	10	2.3	1.8	.....	2.3	....	A6n	.....	.....
28	25 CVn	27	6	2.1	2.2	.....	1.5	....	A3n	.....	.....
29	23 UMa	37	6	1.6	1.9	.....	2.2	....	A4n	.....	.....
30	γ Boo	.....	.....	.....	1.6	.....	1.9	....	A5n	.....	.....
31	γ Her	0.015	.010	−0.3	1.5	.....	1.4	....	A6n	.....	.....
					Harvard F2						
Width II:											
32	81 Leo	.....	.....	.....	3.0	.....	2.5	....	F2	.....	.....
33	53 Vir	0".013	.011	0.7	3.0	.....	3.4	....	F5	.....	.....
34	π Sag	0.015	.013	−1.1	−0.5	.....	−0.7	....	cF3	.....	.....

TABLE XVIII—*Continued*

No.	Star	Trigonometric			Spectroscopic Absolute Magnitude				Spectral Type		
		π	P.E.	Mag.	M.W.	Vic.	N.L.O.	Arc.	M.W.	Vic.	Arc.
		Harvard F5									
Width I:											
35	40 Leo	0".051	.011	3.5	3.3	.....	3.4	....	F5	....	.....
36	δ Equ	0.060	.004	3.5	4.0	4.0	4.1	3.2	F3	F6	F5n
37	ι Peg	0".077	.006	3.4	3.6	3.4	3.5	3.2	F3	F4	F5n
38	γ Ser	78	7	3.3	3.3	4.2	4.2	....	F5	F6	.....
39	110 Her	51	7	2.8	3.1	3.9	3.2	....	F3	F4	.....
40	ξ Peg	49	7	2.8	3.8	3.2	3.3	2.4	F3	F5	F5s
41	α CMn	291	4	2.8	3.1	3.0	3.1	....	F3	F5	.....
42	ω And	24	6	1.9	2.8	3.2	2.8	....	F2	F3	.....
43	35 Dra	22	7	1.8	2.9	.....	3.4	....	F5	....	.....
44	35 Cyg	.....	....	.....	−1.3	−2.7	−1.2	....	cF5	F7	.....
45	41 Cyg	4	5	−1.4	−0.9	−0.8	−1.0	....	cF4	F5	F5n
46	89 Her	.....	....	.....	−1.5	.....	−2.5	....	cF5	....	.....
47	α Per	9	4	−2.0	−1.2	−0.6	−1.5	....	cF4	F4	.....
48	φ Cas	− 6	5	−4.0	−2.8	.....	−3.2	....	cF7	....	.....
49	ε Aur	2	7	−4.0	−1.7	−2.8	−2.8	....	cF2	F2	.....
Width II:											
50	B.S. 3579	72	6	3.4	3.3	3.4	3.3	....	F2	F5	.....
51	τ Boo	61	7	3.4	3.5	4.2	2.9	....	F6	F7	.....
52	ι Vir	42	7	2.3	3.0	.....	2.3	....	F5	....	.....
53	μ Vir	45	5	2.2	2.7	2.0	2.5	....	F3	F3	.....
54	β Cas	71	7	1.7	2.2	1.7	1.7	....	F2	F2	.....
55	β Del	27	6	0.9	2.6	1.9	1.9	2.4	F3	F3	F2n
56	ν Per	16	7	−0.1	−1.0	0.0	0.7	....	cF4	F3	.....
Width III:											
57	α Tri	51	8	2.1	2.4	1.7	1.4	....	F2	F2	.....
58	ξ Gem	48	7	1.8	2.2	0.8	1.9	2.4	F3	F5	F2n
59	κ Peg	0.024	.005	1.2	3.0	2.4	2.8	2.4	F2	F2	A9n
		Harvard F8									
Width I:											
60	η Cas	0".182	.005	4.9	4.8	5.2	4.4	3.6	F9	F8	F7n
61	χ′ Ori	.104	.006	4.7	4.0	4.6	4.4	....	F9	G1	.....
62	γ Lep	.122	.004	4.2	3.8	.....	3.9	....	F6	....	.....
63	χ Dra	.119	.008	4.1	4.0	4.3	4.0	....	F5	F5	.....
64	BS 244	.064	.010	4.0	3.9	.....	3.7	....	F8	....	.....
65	π^3 Ori	.128	.005	3.8	3.5	4.0	3.5	....	F5	F5	.....
66	β Vir	.101	.007	3.8	4.0	3.7	3.5	....	F8	F8	.....
67	θ Per	.078	.005	3.7	3.6	.....	4.1	....	F5	F8	.....
68	θ UMa	0.056	.006	2.0	3.3	2.4	2.9	....	F4	F7	.....

TABLE XVIII—*Continued*

No.	Star	Trigonometric			Spectroscopic Absolute Magnitude				Spectral Type		
		π	P.E.	Mag.	M.W.	Vic.	N.L.O.	Arc.	M.W.	Vic.	Arc.
		Harvard F8—*Continued*									
Width I:											
69	ρ Cas	0".022	.005	1.4	−3.2	.....	−3.5	....	cGo	....	.....
70	11 Pup	.023	.009	1.2	−2.6	.....	.....	....	cG2	....	.....
71	45 Dra	.008	.006	−0.5	−2.0	.....	−2.9	....	cF8	....	.....
72	α UMn	.003	.004	−3.1	−2.2	−2.2	−2.2	....	cF7	F7	.....
73	δ CMa	− .019	.009	−4.0	−3.8	.....	−3.1	....	G3	....	.....
74	γ Cyg	− .004	.007	−4.0	−2.1	−3.2	−3.3	....	cF7	F7	.....
Width II:											
75	θ Boo	.068	.007	3.2	3.3	3.6	3.3	....	F6	F6	.....
76	θ Dra	.046	.005	2.4	3.0	2.8	3.2	....	F7	F7	.....
77	ϵ Hyd	0.025	.006	0.5	2.7	.....	.....	....	F8	....	.....
		Harvard Go									
Width I:											
78	Sun	.....	....	4.8	.....	.....	.....	....	.....	.....	.....
79	λ Aur	0".068	.005	4.0	4.4	5.6	4.4	....	Go	G1	.....
80	ι Per	.085	.004	3.8	3.9	4.3	4.4	....	G1	G1	.....
81	ζ Her	0.110	.005	3.2	3.7	3.2	3.0	....	Go	Go	.....
82	η Boo	.....	....	.....	3.2	3.4	2.5	....	F7	Go	.....
83	υ And	0".061	.007	3.1	3.7	4.0	3.8	....	Go	F8	.....
84	β Cam	14	11	−0.1	−2.5	−3.0	− 3.2	....	cG2	Go	.....
85	α Aur	71	4	−0.5	0.1	1.1	0.1	....	G1	G1	.....
86	υ UMa	5	6	−2.0	0.5	.....	−0.5	....	G1	....	.....
87	43 Gem	6	6	−2.1	.....	−3.2	−3.3	.....	.....	Go	.....
88	β Lep	9	6	−2.3	0.5	.....	−0.6	....	G1	....	.....
89	ϵ Leo	2	8	−2.4	−1.3	.....	−0.9	....	cG3	....	.....
90	β Dra	3	6	−2.5	−1.7	−2.0	.....	....	cG2	Go	.....
Width II:											
91	α Aqr	4	5	−2.3	−2.2	−1.8	−2.5	....	cG1	F9	.....
92	η Peg	−0.002	.007	−2.4	0.6	0.4	−0.5	....	G2	G1	.....

error of the trigonometric parallax; (5) the absolute magnitude derived from the trigonometric parallax; (6), (7), (8), and (9) the spectroscopic parallaxes determined at Mount Wilson, Victoria, the Norman Lockyer Observatory, and Arcetri; (10), (11), and (12) the spectral classes determined at Mount Wilson, Victoria, and Arcetri.

Both Tables XVIII and XIX group the stars according to Har-

TABLE XIX

No.	Star	K	$H\delta$	λ 4045	λ 4063	λ 4071	λ 4077	λ 4132	λ 4161	λ 4167	λ 4215	λ 4226	λ 4246	λ 4250	λ 4233	λ 4235
		Harvard Fo														
Width I:																
1.....	σ Boo	4.4	6.3	0.50	0.37	0.30	0.28	0.27	0.15	0.28	0.35	0.50	0.20	0.32	0.32	0.25
2.....	γ Vir(S)	5.0	6.6	.57	.50	.35	.35	.20	.20	.35	.30	.47	.18	.28	.26	.20
3.....	γ Vir(N)	5.0	6.6	.52	.49	.33	.38	.20	.18	.29	.33	.46	.27	.32	.29	.26
4.....	9 Aur	4.3	.....	.59	.45	.32	.38	.23	.25	.38	.44	.65	.35	.33	.39	.25
5.....	η Lep	4.3	.....	.57	.42	.33	0.40	.23	.27	.36	.45	.63	.30	.38	.42	.34
6.....	γ Equ	.....	.....	.55	.40	.18	1.50	.29	.26	.50	.90	.50	.28	.25	.28	.27
7.....	β CrB	4.4	9.0	.50	.35	.18	0.30	.41	.33	.23	.60	.40	.28	.30	.32	.22
8.....	20 CVn	6.0	....	.60	.52	.32	.42	.57	.33	.44	.48	.60	.40	.41	.47	.35
9.....	ν Her	5.0	6.3	.68	.42	.32	.44	.27	.34	.22	.52	.65	.33	.40	.55	.33
10.....	α Lep	7.3	7.0	.65	.59	.32	.60	.40	.50	.28	.56	.56	.47	.35	.64	.30
Width II:																
11.....	η Crv	6.2	.....	.70	.45	.40	bl	.21	.30	.42	.42	.70	.22	.30	.40	.29
12.....	ρ Gem	7.7	7.3	.58	.41	.32	bl	.24	.28	.33	.37	.62	.33	.30	.35	.28
13.....	7 And	4.0	8.9	.55	.53	.30	bl	.25	.22	.32	.35	.48	.32	.35	.34	.24
14.....	18 Boo	.....	.....	.70	.42	.38	.32	.27	.27	.38	.45	.72	bl	.34	.35	.33
15.....	ω Oph	3.5	.....	.40	.33	.18	.42	.14	.33	.24	.40	.30	.10	.15	.33	.27
16.....	30 LMi	4.5	.....	.69	.39	.30	.33	.38	.30	.32	.50	.58	.35	.38	.46	.33
17.....	μ Cet	4.0	5.5	.50	.45	.40	bl	.33	.20	.32	.38	.60	.34	.37	.36	.25
18.....	γ Cap	3.5	.....	.64	.54	.30	.80	.23	.42	.24	.60	.45	.30	.28	.36	.33
19.....	22 And	4.4	6.5	.55	.48	.38	.50	.28	0.30	0.20	0.40	0.45	0.38	0.34	0.38	0.22
Width III:																
20.....	16 Lib	4.9	8.1	.46	.36	.19	.82	.22	.....	.....	.....	.....	.....	.....	.....	.....
21.....	38 Gem	5.5	6.3	.25	.16	.14	.42	.15	.....	.....	.....	.....	.....	.....	.....	.....
22.....	τ Cyg	5.9	4.1	.50	.34	.20	.57	.25	.....	.....	.....	.....	.....	.....	.....	.....
23.....	δ Aql	5.2	6.5	.43	.33	.72	.59	.19	.....	.....	.....	.....	.....	.....	.....	.....
24.....	ο Gem	4.6	3.2	.33	.21	.18	.48	.20	.....	.....	.....	.....	.....	.....	.....	.....
25.....	ζ Leo	5.3	5.9	0.55	0.45	0.46	0.69	0.33	.....	.....	.....	.....	.....	.....	.....	.....

TABLE XIX—*Continued*

No.	Star	K	*Hδ*	λ 4045	λ 4063	λ 4071	λ 4077	λ 4132	λ 4161	λ 4167	λ 4215	λ 4226	λ 4246	λ 4250	λ 4233	λ 4235
							Harvard Fo—*Continued*									
Width IV:																
26	δ Gem	5.2	9.2	0.35	0.33	0.14	0.55	0.19	.....	.....	.....	.....	.....	.....	.....	.....
27	78 UMa	6.5	5.5	.40	.26	.31	.59	.23	.....	.....	.....	.....	.....	.....	.....	.....
28	25 CVn	4.2	8.4	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
29	23 UMa	4.7	7.0	.31	.25	.....	.53	.....	.....	.....	.....	.....	.....	.....	.....	.....
30	γ Boo	6.3	11.5	.43	.44	.40	.74	.....	.....	.....	.....	.....	.....	.....	.....	.....
31	γ Her	5.7	8.9	0.31	0.23	0.12	0.53	0.10	.....	.....	.....	.....	.....	.....	.....	.....
							Harvard F2									
Width II:																
32	81 Leo	5.5	.....	0.62	0.42	0.35	0.45	0.22	0.34	0.32	0.45	0.70	0.32	0.36	0.42	0.35
33	53 Vir	6.2	.....	.85	.42	.40	.42	.43	.30	.35	.47	.75	.22	.40	.42	.36
34	π Sag	6.0	4.3	0.60	0.58	0.35	0.50	0.39	0.38	0.34	0.44	0.50	0.38	0.36	0.46	0.33
							Harvard F5									
Width I:																
35	40 Leo	3.7	.....	0.77	0.55	0.40	0.38	0.35	0.32	0.38	0.40	0.70	0.33	0.39	bl	0.41
36	δ Equ	4.3	.....	.77	.72	.42	.32	.29	.26	.35	.33	.80	.20	.37	.32	.29
37	ι Peg	5.6	3.0	.60	.50	.40	.38	.26	.32	.38	.36	.55	.24	.28	.32	.34
38	γ Ser	6.0	3.3	.66	.71	.40	.32	.21	.15	.30	.32	.65	.17	.26	.28	.27
39	110 Her	6.3	4.4	0.64	0.55	0.46	0.35	0.31	0.28	0.38	0.40	0.60	0.31	0.33	0.33	0.30

TABLE XIX—*Continued*

No.	Star	K	*Hδ*	λ 4045	λ 4063	λ 4071	λ 4077	λ 4132	λ 4161	λ 4167	λ 4215	λ 4226	λ 4246	λ 4250	λ 4233	λ 4235
		Harvard F5—*Continued*														
Width I:																
40.....	ξ Peg	3.7	.....	0.83	0.63	0.46	0.42	0.23	0.28	0.44	0.46	0.65	0.28	0.33	0.45	0.36
41.....	α CMi	6.3	4.4	.70	.54	.38	.41	.31	.33	.37	.45	.63	.24	.30	.36	.32
42.....	ω And	6.0	3.6	.65	.68	.40	bl	.33	.28	.36	.40	.75	.31	.32	.34	.31
43.....	35 Dra	4.3	.....	.72	.63	.50	.46	.46	.32	.36	.50	.66	.28	.32	.38	.37
44.....	35 Cyg	.....	.....	.80	.53	.38	.53	.62	.48	.28	.80	.68	.46	.50	.50	.41
45.....	41 Cyg	6.9	.....	.70	.55	.45	.60	.46	.46	.28	.60	.60	.45	.42	.58	.40
46.....	89 Her	.....	.....	.71	.72	.45	.55	.35	.48	.25	.75	.45	.45	.45	.72	.34
47.....	α Per	7.6	6.0	.76	.58	.48	.62	.46	.54	.32	.60	.65	.44	.44	.56	.40
48.....	φ Cas	4.4	.....	.60	.52	.50	.72	.31	.55	.15	.55	.52	.57	.33	.60	.30
49.....	ε Aur	4.4	3.0	.70	.66	.45	.78	.33	.48	.10	.68	.58	.75	.23	.80	.18
Width II:																
50.....	BS 3579	5.5	3.3	.70	.50	.42	.36	.43	.32	.42	.50	.75	.30	.35	.37	.33
51.....	τ Boo	5.2	.....	.80	.52	.48	.36	.49	.35	.45	.45	.80	.35	.41	.40	.43
52.....	ι Vir	4.0	.....	.68	.58	.45	.54	.28	.33	.32	.45	.65	.28	.29	.35	.33
53.....	μ Vir	5.1	5.8	.64	.42	.35	.54	.23	.24	.32	.37	.64	.25	.24	.29	.23
54.....	β Cas	5.0	7.5	.45	.51	.46	.70	.33	.18	.19	.46	.90	.55	.42	.....	.....
55.....	β Del	6.0	4.6	.72	.48	.48	bl	.33	.31	.37	.40	.65	.32	.33	.34	.32
56.....	ν Per	6.7	.....	.73	.45	.40	0.70	.63	0.45	0.26	0.72	0.75	0.42	0.45	0.60	0.52
Width III:																
57.....	α Tri	4.3	6.1	.69	.61	.43	1.20	.50	.....	.....	.....	.....	.....	.....	.....	.....
58.....	ξ Gem	5.7	6.1	.91	.70	.60	1.20	.39	.....	.....	.....	.....	.....	.....	.....	.....
59.....	κ Peg	5.2	.69	0.69	0.52	0.33	1.22	0.41	.....	.....	.....	.....	.....	.....	.....	.....

TABLE XIX—*Continued*

No.	Star	K	*H* δ	λ 4045	λ 4063	λ 4071	λ 4077	λ 4132	λ 4161	λ 4167	λ 4215	λ 4226	λ 4246	λ 4250	λ 4233	λ 4235
		Harvard F8														
Width I:																
60	η Cas	6.0	.....	1.1	0.97	0.67	0.32	0.75	0.20	0.38	0.40	1.4	0.28	0.40	0.34	0.37
61	χ^1 Ori	6.6	.....	1.1	0.90	0.85	0.35	0.75	0.24	.37	0.48	1.6	0.26	0.35	0.35	.37
62	γ Lep	5.5	.....	0.95	0.73	0.55	0.46	0.50	0.30	.39	0.45	1.1	0.27	0.35	0.40	.42
63	χ Dra	6.6	3.5	1.0	0.75	0.52	0.39	0.36	0.15	.37	0.42	1.1	0.20	0.30	0.35	.32
64	BS 244	.....	.....	1.0	0.83	0.60	0.40	0.40	0.29	.38	0.55	1.2	0.30	0.48	0.50	.53
65	π^3 Ori	6.2	4.3	0.50	0.45	0.40	0.38	0.25	0.20	.30	0.38	0.60	0.25	0.38	0.38	.33
66	β Vir	7.7	3.1	0.95	0.76	0.60	0.47	0.46	0.27	.48	0.48	0.80	0.26	0.50	0.50	.53
67	θ Per	.....	.....	0.95	0.90	0.60	0.45	0.60	0.29	.38	0.45	0.90	0.27	0.45	0.40	.35
68	θ UMa	6.7	1.8	0.60	0.45	0.40	0.35	0.29	0.28	.30	0.35	0.50	0.28	0.30	0.32	.27
69	ρ Cas	9.0	.....	1.8	1.6	1.3	2.2	1.0	1.2	.15	2.0	1.6	1.9	1.0	1.7	.65
70	11 Pup	3.4	.....	1.0	0.75	0.50	0.90	0.68	0.55	.45	0.95	1.0	0.55	0.58	0.77	.50
71	45 Dra	7.3	.....	0.80	0.70	0.40	0.75	0.62	0.62	.30	0.70	0.70	0.65	0.65	0.75	.50
72	α UMi	7.3	3.3	0.70	0.50	0.38	0.65	0.44	0.45	.25	0.70	0.70	0.58	0.45	0.50	.30
73	δ CMa	9.0	1.7	1.5	1.4	0.90	1.9	1.1	1.2	.37	1.5	1.4	1.3	1.1	1.2	.90
74	γ Cyg	7.6	.....	0.90	0.85	0.45	0.85	0.64	0.70	.39	0.95	0.95	0.80	0.70	0.75	.75
Width II:																
75	θ Boo	5.5	.....	0.90	0.65	0.50	0.38	0.33	0.34	.47	0.45	1.20	0.30	0.38	0.42	.40
76	θ Dra	5.0	.....	0.85	0.60	0.50	0.36	0.39	0.33	.47	0.52	0.85	0.28	0.42	0.45	.40
77	ε Hyd	.....	.....	0.90	0.75	0.50	0.37	0.43	0.12	0.37	0.60	1.20	0.15	0.30	0.38	0.35

TABLE XIX—*Continued*

No.	Star	K	$H\delta$	λ 4045	λ 4063	λ 4071	λ 4077	λ 4132	λ 4161	λ 4167	λ 4215	λ 4226	λ 4246	λ 4250	λ 4233	λ 4235
		Harvard Go														
Width I:																
78.....	Sun	7.4	1.0	1.4	1.0	0.95	0.45	0.73	0.18	0.55	0.45	1.8	0.27	0.48	0.38	0.43
79.....	λ Aur	6.9	1.6	1.4	1.05	.80	.35	.40	.20	.46	.55	1.5	.23	.45	.33	.40
80.....	ι Per	7.2	2.5	1.4	0.92	.66	.50	.46	.08	.51	.55	1.4	.20	.29	.45	.48
81.....	ζ Her	6.9	1.6	1.2	0.80	0.65	0.37	0.31	0.20	0.39	0.46	1.1	0.30	0.34	0.34	0.36
82.....	η Boo	6.9	3.2	1.2	0.65	0.67	0.37	0.43	0.25	0.65	0.80	1.2	0.18	0.35	0.55	0.60
83.....	υ And	6.6	2.5	1.2	0.77	0.65	0.38	.55	.28	.45	0.51	1.1	0.25	.45	0.40	0.39
84.....	β Cam	6.6	.....	1.6	1.1	0.70	1.0	.80	.85	.58	1.2	1.7	1.0	.70	1.2	1.0
85.....	α Aur	6.6	1.4	1.0	0.77	0.68	0.55	.40	.26	.40	0.66	1.0	0.20	.29	0.32	0.31
86.....	ο UMa	7.3	.....	1.6	1.2	1.0	0.94	.92	.35	.60	1.4	1.6	0.25	.61	0.60	0.55
87.....	43 Gem	7.4	.....	1.1	0.65	0.58	0.90	.50	.65	.28	0.85	1.0	0.58	.40	0.68	0.40
88.....	β Lep	7.7	.....	1.3	0.81	0.65	0.68	.57	.45	.57	0.90	1.3	0.40	.48	0.50	0.47
89.....	ε Leo	7.7	.....	1.2	0.81	0.65	0.80	.67	.37	.40	1.20	1.3	0.33	.40	0.58	0.50
90.....	β Dra	6.0	1.2	1.1	0.79	0.67	0.90	.50	.48	.44	1.00	1.0	0.34	.44	0.58	0.50
Width II:																
91.....	α Aqr	8.1	.....	1.7	1.05	0.88	1.5	.76	.65	.53	1.5	1.9	0.50	.42	0.80	0.70
92.....	η Peg	8.1	.....	1.5	0.88	0.92	1.1	0.60	0.54	0.65	1.5	1.9	0.18	0.40	0.60	0.58

vard spectral classes; within each class the stars are arranged according to line width and within each group according to increasing luminosity determined from the trigonometric parallaxes.

Table XIX gives in successive columns the serial number; the star's name; the intensities in angstroms of total absorption of *Ca* II K, *Hδ*, *Fe* I 4045, *Fe* I 4063, *Fe* I 4071, *Sr* II 4077, *Fe* I 4132, *Zr* II—*Ti* II 4161, *Mg* I 4167, *Sr* II 4215, *Ca* I 4226, *Sc* II 4246, *Fe* I 4250, *Fe* II—*Fe* I 4233, and *Fe* I 4235.

In addition to the topics discussed in the present paper, a number of other phases of the F-type spectra are under investigation. A preliminary report on some of these has already been published.[11] These include investigations of the composite stars, the behavior of the *CH* molecule with respect to spectral class and luminosity, and a general discussion of the question of the classification of the F stars. It is expected that the greater part of the work will be completed in a short time.

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REVIEWS

The Binary Stars. By ROBERT GRANT AITKEN. 2d ed. New York: McGraw-Hill Book Co., 1935. Pp. xii+309; Figs. 16; Pls. 4. Bound, $3.75.

In this new edition the author has maintained the same arrangement of the subject matter that was used in the first edition. A great deal of new material has replaced previous sections and has added to the usefulness of the book without increasing its size. The same collaborator, J. H. Moore, has contributed a revised chapter on radial velocities, while the section on eclipsing binaries follows closely the latest methods developed at the Princeton Observatory. The statistical results have been reinforced through the addition of many more observational data, but the earlier conclusions have not been modified. From the theoretical point of view there have been a large number of contributions to the subject since the first edition was published in 1918. The author mentions these only in part. Their inclusion would have required an appreciable enlarging of the volume.

The reader is glad to find the Thiele-Innes method for deriving an orbit included in a new paragraph, but beginners will be disturbed by a number of typographical errors. The double area of the triangle (p. 92, l. 3) should be $x_p y_q - x_q y_p$ instead of $X_p Y_q - X_q Y_p$. In equations (45*a*) the three areas should be $\Delta_{1,2}$, $\Delta_{2,3}$, and $\Delta_{1,3}$ instead of $\Delta_{1,2}$, $\Delta_{3,2}$, and $\Delta_{3,1}$, which means a change of sign in the last two. In equation (48) the last term of both denominators should be $\Delta_{1,3}$ instead of $\Delta_{3,1}$, and the sign of the last term of the numerator in the second equation should be $-$ instead of $+$; also a few lines farther E_3 follows from E_2+v instead of from E_2-v.

In redrawing Figure 6, which was Figure 4 in the first edition, a number of letters have been omitted, which makes it difficult to follow the text of page 109. In the construction of the point S' the author inadvertently makes the distances OS' and OS in the ratio of the cosine of the inclination, instead of taking the distances from the nodal line.

Of special interest are the tables of elements for both the visual and the spectroscopic binaries given in the Appendix. That the numbers of these have increased from 87 and 137 to 116 and 326 in the new volume illustrates the rapid growth of the data that have become available since 1918.

G. V. B.